AN EXTENSION OF A COVARIANT DIFFERENTIATION PROCESS

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Craig\(^2\) has considered tensors \(T^\alpha_\beta; \ldots\) whose components are functions of \(n\) variables represented by \(x\) and their \(m\) derivatives \(x', x'', \ldots, x^{(m)}\). He obtained the covariant derivative

\[
T^\alpha_\beta; \ldots x^{(m)}y - mT^\alpha_\beta; \ldots x^{(m)}\gamma\{^\lambda_\gamma\}, \quad m \geq 2,
\]

where

\[
\{^\lambda_\gamma\} = \frac{\partial}{\partial y^\gamma} \Gamma^\lambda_\gamma + (1/2) R^\beta_\gamma x^\alpha x^\beta \delta^\lambda_\alpha,
\]

and partial differentiation in (1) is denoted by the added subscript. Throughout, a repeated letter in one term indicates a sum of \(n\) terms. The purpose of this note is to derive another tensor from \(T^\alpha_\beta; \ldots\) whose covariant rank is one larger. The general process will be shown clearly by using \(T^\alpha(x, x', x'', x''')\).

The extended point transformation

\[
x^\alpha = x^\alpha(y), \quad x'^\alpha = \frac{\partial x^\alpha}{\partial y^i} y^i,
\]

\[
x''^\alpha = \frac{\partial x^\alpha}{\partial y^i} y'^i + \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^i y'^j + \cdots, \quad \alpha = 1, \ldots, n,
\]

gives the tensor equations of transformation of the tensor \(T^\alpha\) as

\[
\bar{T}^i(y, y', y'', y''') = T^\alpha(x, x', x'', x''') \frac{\partial y^i}{\partial x^\alpha},
\]

where \(y\) stands for the \(n\) variables \(y^1, y^2, \ldots, y^n\) and a similar notation is used for the derivatives \(y', y'', y'''\). On differentiating equations (3) with respect to \(y'^k\) it is found that

\[
\bar{T}_y^{i,k} = \left( T^\alpha_\beta \frac{\partial x^\beta}{\partial y^k} + T_z^\alpha_\beta \frac{\partial x'^\beta}{\partial y'^k} + T_x^\alpha_\beta \frac{\partial x''^\beta}{\partial y''^k} + T_y^\alpha_\beta \frac{\partial x'''^\beta}{\partial y'''^k} \right) \frac{\partial y^i}{\partial x^\alpha}.
\]

The derivatives can be expressed by using the following general formulas:

\footnote{1} Presented to the Society, April 15, 1939.
\[
\frac{\partial x^{(m-1)\beta}}{\partial y^{(m-2)k}} = (m - 1) \frac{\partial x^{\beta}}{\partial y^k}, \quad \frac{\partial x^{(m)\beta}}{\partial y^{(m-2)k}} = \frac{m(m - 1)}{2} \frac{\partial x^{''\beta}}{\partial y^k},
\]
in which \(\partial x^{''\beta}/\partial y^k\) are eliminated by

\[
\{\beta_k^i\} \frac{\partial x^\beta}{\partial y^l} = \frac{\partial x^{''\beta}}{\partial y^k} + \{\beta_\gamma^i\} \frac{\partial x^\gamma}{\partial y^k}.
\]

The derivatives \(\partial x^{''\beta}/\partial y^k\) are simplified by first writing

\[
x^{''\beta} = \frac{\partial x^\beta}{\partial y^i} y^{''i} + \Gamma^r_{ik} y^{''i} \frac{\partial x^\beta}{\partial y^r} - \Lambda^\beta_{ai} x^{''i} x^a,
\]
with the help of (2), (6) and \(f_{\alpha\beta\gamma}x^{''\beta} = 0\). It is necessary also to have

\[
\frac{\partial^2 x^\beta}{\partial y^i \partial y^k} = \Lambda^t_{jk} \frac{\partial x^\beta}{\partial y^i} - \Lambda^\beta_{ai} \frac{\partial x^a}{\partial y^i} \frac{\partial x^\beta}{\partial y^k},
\]
where

\[
\Lambda^\beta_{ai} = \Gamma^\beta_{ai} - \frac{1}{2} f_{\gamma\tau} (f_{\beta\gamma} \{\tau^i\} + f_{\gamma\tau\rho} \{\rho^i\} - f_{\beta\tau\rho} \{\rho^i\}).
\]

This is obtained from Taylor’s formula (19) in the following way. Multiply this formula by \((\partial y^k/\partial x^\tau)f^\beta_{\tau i} = (\partial x^\beta/\partial y^i)f^\beta_{\tau i},\) and sum for \(k\). Use the tensor equations for \(f_{\alpha\beta\gamma}\) and substitute from (6) for \(\partial x^{''\beta}/\partial y^i\).

By means of formulas (6) and (8) and the tensor \(Q^\beta(x, x', x'') = x^{''\beta} + \Gamma^\beta_{ai} x^{''i} x^a\) the partial derivatives of (7) have the form

\[
\frac{\partial x^{''\beta}}{\partial y^k} = -\{\beta_\gamma^i\} \frac{\partial x^\gamma}{\partial y^k} + \frac{\partial x^\beta}{\partial y^k} - 2\{\beta_\alpha^i\} \{\tau^i\} \frac{\partial x^\alpha}{\partial y^i} + 2\{\beta_\gamma^i\} \{\tau^i\} \frac{\partial x^\beta}{\partial y^i} + 2\{\gamma^i\} \{\tau^i\} \frac{\partial x^\gamma}{\partial y^i} - 2\{\beta_\alpha^i\} \{\tau^i\} \frac{\partial x^\alpha}{\partial y^i},
\]
in which we have the nontensor form

\[
|\beta_\gamma^i| = Q_{\gamma\beta} - Q_{\gamma\alpha} \{\gamma^i\} + Q_{\beta\tau} \Lambda^\beta_{\alpha\gamma}.
\]

If formulas (6) and (9) are substituted in equations (5) and the results used in (4), we find

\[
\overline{T}^{il}_{y^k} = (T^\alpha_{x^\beta} - 2T^\alpha_{x^\beta} \{\beta^i\} - 3T^\alpha_{y^i} \{\beta^i\} \{\delta^\beta\} \{\delta^\beta\} \{\beta^i\}) \frac{\partial x^\beta}{\partial y^i} \frac{\partial y^i}{\partial x^a} - (- 2\overline{T}^{il}_{y^i} \{\beta^i\} - 3\overline{T}^{il}_{y^i} \{\beta^i\} \{\beta^i\} \{\beta^i\}).
\]

Hence the new tensor whose covariant rank has been increased by one is

\begin{equation}
T^\alpha_{\varepsilon} - 2 T^\alpha_{\varepsilon'} \delta^\delta_{\varepsilon'} - 3 T^\alpha_{\varepsilon'' \varepsilon} \delta^\delta_{\varepsilon'' \varepsilon'},
\end{equation}

where \( \delta^\delta_{\varepsilon'} \) and \( \delta^\delta_{\varepsilon'' \varepsilon} \) are defined in (2) and (10).

Because of the general relations in (5) it is easy to verify that the tensor

\begin{equation}
T^{\alpha \cdots \varepsilon (m-2)}_{\varepsilon} - (m - 1) T^{\alpha \cdots \varepsilon (m-3)}_{\varepsilon'} \delta^\delta_{\varepsilon'} - \frac{m(m - 1)}{2} T^{\alpha \cdots \varepsilon (m)}_{\varepsilon''} \delta^\delta_{\varepsilon''},
\end{equation}

has a covariant rank which is one larger than that of \( T^{\alpha \cdots \varepsilon} \), whose components are functions of \( (x, x', \ldots, x^{(m)}) \).

If the components of the tensor \( T^\alpha(x, x', x'', x^{''''}) \) do not contain the derivatives \( x^{''''} \), then (11) reduces to Craig's covariant derivative (1), and if there are no \( x'' \) or \( x^{''''} \) derivatives, then the result is a partial differentiation with respect to \( x' \).

The usual rules for the derivative of a sum of tensors of the same type and rank and for the product of any tensors are preserved by this process.

If \( m = 2 \), a scalar \( T(x, x', x'') \) will give a covariant tensor which is similar to that in (11) when the tensor equations for \( T(y, y', y'') \) are differentiated with respect to \( y \) instead of \( y' \). The tensor is

\begin{equation}
T^\alpha_{\varepsilon} - T^\alpha_{\varepsilon'} \delta^\delta_{\varepsilon'} - T^\alpha_{\varepsilon'' \varepsilon} \delta^\delta_{\varepsilon'' \varepsilon'}.
\end{equation}

However, if \( m = 2 \) and a tensor \( T^\alpha(x, x', x'') \) is used, an extra term \( T^\delta \Lambda^\alpha_{\varepsilon} \) has to be added to three terms similar to those in (13). If this process is performed on the tensor \( Q^\alpha(x, x', x'') \), the result is the zero tensor.