A DECOMPOSITION OF ADDITIVE SET FUNCTIONS

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This paper is concerned with a decomposition theorem for additive functions on an additive family of sets to either real numbers or a Banach space. Additive bounded set functions have as yet been little studied. However the recent paper of Hildebrandt\(^2\) illustrates their importance.

We shall use the following notation:

(a) \(T\): an abstract class of arbitrary elements \(t\).

(b) \(\mathcal{J}\): a completely additive family of subsets \(\tau\) of \(T\); that is, \(T \in \mathcal{J}\), \(\tau \in \mathcal{J}\) implies \(T - \tau \in \mathcal{J}\), and \(\tau_n \in \mathcal{J}\) for \(n = 1, 2, \ldots\) implies \(\sum \tau_n \in \mathcal{J}\).

(c) \(\alpha\): a set function on \(\mathcal{J}\) to real numbers.

(d) \(A\): the subclass of set functions on \(\mathcal{J}\) to real numbers which are additive and bounded; that is, \(\tau_1, \tau_2 \in \mathcal{J}\) and \(\tau_1 \cdot \tau_2 = 0\) implies \(\alpha(\tau_1 + \tau_2) = \alpha(\tau_1) + \alpha(\tau_2)\).

(e) \(C\): the subclass of set functions on \(\mathcal{J}\) to real numbers which are completely additive (c.a.), that is, \(\tau_n \in \mathcal{J}\) for \(n = 1, 2, \ldots\) and \(\tau_i \cdot \tau_j = 0\) if \(i \neq j\) implies \(\alpha(\sum \tau_n) = \sum \alpha(\tau_n)\). The functions in \(C\) are bounded.\(^3\)

The notations \(A_P\) and \(C_P\) refer to the subclasses of \(A\) and \(C\) respectively whose elements are nonnegative.

(f) \(x\): a set function on \(\mathcal{J}\) to a Banach space\(^4\) \(X\). The definitions of additive and c.a. set functions are formally retained. If \(\{\tau_n\}\) is a sequence of disjoint sets of \(\mathcal{J}\) and \(x(\tau)\) is c.a., then \(\sum x(\tau_n)\) is unconditionally convergent.\(^5\)

(g) \(C_X\): the class of c.a. set functions on \(\mathcal{J}\) to \(X\).

In the statement of the following theorems, \(D\) will designate any one of the classes \(A, A_P, C, C_P, \) and \(\mathcal{F}\) will denote the cardinal number of \(\tau\).

**Theorem 1.** Let \(\mathbb{N}\) be an infinite cardinal number not greater than \(\overline{T}\). For every \(\alpha \in D\) there exists an unique decomposition \(\alpha = \alpha_1 + \alpha_2\) and a set \(R(\alpha) \in \mathcal{J}\) of cardinal number not greater than \(\mathbb{N}\) such that \(\alpha_1, \alpha_2 \in D\),

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\(^1\) Presented to the Society April 15, 1939, under the title On additive set functions.


\(^4\) S. Banach, Théorie des Opérations Linéaires, Monografje Matematyczne, Warsaw, 1932, chap. 5.

\(^5\) If \(x_n\) is a series of elements of \(X\) and if every subseries \(\sum x_n\) is convergent, then \(\sum x_n\) is said to be unconditionally convergent.
Let \( \Sigma = E_{\bar{\tau}} \), \( \bar{\tau} \leq \aleph_0 \), \( \alpha(\tau) \neq 0 \). We define a transfinite sequence \( (\tau_1, \tau_2, \cdots ; \tau_w, \cdots, \tau_\lambda, \cdots) \) as follows: \( \tau_1 \) is an arbitrary element of \( \Sigma \). Suppose \( \tau_\lambda \) have been defined for all \( \lambda < \mu \). If there exists \( \tau \) such that \( \tau \cdot \sum_{\lambda < \mu} \tau_\lambda = 0 \) and \( \tau \in \Sigma \), then we set \( \tau = \tau_\mu \).

As \( \alpha(\tau) \) is bounded, \( \alpha(\tau) \) cannot differ from zero on a nondenumerable number of disjoint sets. The sequence therefore contains only a denumerable set of elements.

Let \( R = \sum \lambda \tau_\lambda \). Then \( R \in \mathfrak{A} \) and \( \bar{R} \leq \aleph_0 \). We define \( \alpha_1(\tau) = \alpha(R \cdot \tau) \), \( \alpha_2(\tau) = \alpha(\tau) - \alpha_1(\tau) = \alpha(\tau - R \cdot \tau) \). The \( \alpha_1(\tau) \), \( \alpha_2(\tau) \) are clearly elements of \( D \). If \( \bar{\tau} \leq \aleph_0 \), then by the definition of \( R \), \( \alpha(\tau) = \alpha(\tau - R \cdot \tau) = 0 \).

Although the set \( R \) is not unique, the function decomposition is unique: Suppose there exist two different sets \( R_1, R_2 \) having the properties of the \( R \) defined above. The set identity \( R_1 \cdot \tau + (R_2 - R_1) \cdot \tau = R_2 \cdot \tau + (R_2 - R_1) \cdot \tau \) and \( \alpha[(R_1 - R_2) \cdot \tau] = 0 = \alpha[(R_2 - R_1) \cdot \tau] \) imply that \( \alpha(R_1 \cdot \tau) = \alpha(R_2 \cdot \tau) \).

A set function \( \alpha \) on \( \mathfrak{A} \) will be said to be \textit{nonsingular} if for every \( t \in \mathfrak{A} \), \( \alpha(t) = 0 \). A set function \( \alpha \) on \( \mathfrak{A} \) will be called \textit{\( \aleph_0 \)-homogeneous} if there exists a set \( \tilde{R} \) such that \( \tilde{R} \in \mathfrak{A} \), \( \bar{\tilde{R}} = \aleph_0 \), \( \alpha(\tau) = \alpha(\tilde{R} \cdot \tau) \), and \( \alpha(\tau) = 0 \) if \( \bar{\tau} \leq \aleph_0 \).

Without loss of generality we may consider only nonsingular set functions because for every \( \alpha \in D \) there exists a unique decomposition \( \alpha = \alpha_1 + \alpha_2 \) and a denumerable set \( \{t_i\} \) of elements of \( T \), such that \( \alpha_1, \alpha_2 \in D \), \( \alpha_1(\tau) = \sum_{i=1}^{\infty} \alpha(t_i) \cdot \tau \), and \( \alpha_2 \) is nonsingular. We omit the proof.

**Theorem 2.** For every nonsingular \( \alpha \in D \), there exists an unique decomposition \( \alpha = \sum_i \alpha_i \), the sum being absolutely convergent, and such that \( \alpha_i \) is \( \aleph_0 \)-homogeneous and \( \aleph_0 \neq \aleph_i \); if \( i \neq j \).

In the proof of this theorem an induction is made on the infinite cardinals not exceeding that of \( T \), well-ordered according to magnitude. We define a transfinite sequence of set functions \( (\alpha_1, \alpha_2, \cdots; \alpha_\alpha, \cdots, \alpha_\lambda, \cdots) \) as follows: Suppose \( \alpha_\lambda \) have been defined for all \( \lambda < \mu \) and (1) only a denumerable number of the \( \alpha_\lambda \) are not identically zero; (2) \( \sum_{\lambda \leq \lambda_0} |\alpha_\lambda(\tau)| < \infty \); and (3) \( \alpha_\lambda \in D \) and is \( \aleph_0 \)-homogeneous. By Theorem 1 there exist \( R_\mu \in \mathfrak{A} \) and a decomposition \( \alpha = \alpha_1^1 + \alpha_2^2 \) such that \( \bar{R}_\mu = \aleph_\mu \), \( \alpha_1^1(\tau) = \alpha(R_\mu \cdot \tau) \), \( \alpha_2^2(\tau) = 0 \) if \( \bar{\tau} \leq \aleph_\mu \), and \( \alpha_1^1, \alpha_2^2 \in D \). Clearly \( \alpha_\lambda(\tau) = \alpha(R_\mu \cdot R_\lambda \cdot \tau) \) if \( \lambda < \mu \).

Let \( \alpha_\mu(\tau) = \alpha_1^1(\tau) - \sum_{\lambda < \mu} \alpha_\lambda(\tau) \). We consider the following cases:

I. \( \alpha \in C, C_p \). Let \( \bar{R}_\mu = R_\mu - \sum_{\tau_\lambda} \bar{R}_\lambda \) where \( \pi_\mu = E_{\lambda}[\lambda < \mu, \alpha_\lambda \neq 0] \). The sets \( \bar{R}_\mu \) are disjoint. Suppose \( \alpha_\lambda(\tau) = \alpha(\bar{R}_\lambda \cdot \tau) \) for \( \lambda < \mu \). Then by (1)
\[ \alpha_{\mu}(\tau) = \alpha(R_{\mu} \cdot \tau) - \sum_{\pi_{\mu}} \alpha_{\lambda}(\tau) = \alpha(R_{\mu} \cdot \tau) - \sum_{\pi_{\mu}} \alpha(R_{\mu} \cdot \overline{R}_{\lambda} \cdot \tau) \]

\[ = \alpha \left[ \left( R_{\mu} - \sum_{\pi_{\mu}} R_{\mu} \cdot \overline{R}_{\lambda} \right) \cdot \tau \right] = \alpha(R_{\mu} \cdot \tau) . \]

It is clear that (1), (2), and (3) are satisfied for \( \mu + 1 \). \( \alpha_{\lambda} \neq 0 \) implies that \( \alpha(\tau) \neq 0 \) for some subset of \( \overline{R}_{\lambda} \). As the \( \overline{R}_{\lambda} \) are disjoint, the sequence will contain only a denumerable number of functions not identically zero.

II. \( \alpha \in A_\mu \). For \( \lambda_0 < \mu \), \( \alpha(T) \geq \sum_{\lambda_0} \alpha_{\lambda}(T) \geq \sum_{\lambda_0} \alpha_{\lambda} \). Clearly (1) and (2) are satisfied for \( \mu + 1 \), and the sequence contains only a denumerable number of functions not identically zero. Let \( \lambda_i \) be a spanning sequence for \( E_{\lambda}[\lambda < \mu, \alpha_{\lambda} \neq 0] \). Then

\[ \alpha_{\mu}(\tau) = \alpha_{\lambda}(\tau) - \sum_{\lambda < \mu} \alpha_{\lambda}(\tau) = \alpha(R_{\mu} \cdot \tau) - \lim_{t \to \infty} \alpha_{\lambda}(\tau) \]

\[ = \alpha(R_{\mu} \cdot \tau) - \lim_{t \to \infty} \alpha(R_{\mu} \cdot R_{\lambda_i} \cdot \tau) . \]

Hence (3) is likewise satisfied.

III. \( \alpha \in A \). Every \( \alpha \in A \) has a decomposition \( \alpha = \alpha_1 - \alpha_2 \) where \( \alpha_1, \alpha_2 \in A_\mu \). An application of II to \( \alpha_1 \) and \( \alpha_2 \) gives the desired decomposition.

The decomposition is unique: Any two sequences of homogeneous functions differ in a first function, \( \alpha_{\mu} \). But this is contrary to \( \alpha_{\mu} = \sum_{\lambda \leq \mu} \alpha_{\lambda} \) being unique.

In these theorems the restriction that the additive bounded set function be defined over an additive family \( \mathcal{A} \) is optional, since the range of definition of such a function can always be extended to an additive family. The type of argument used by Pettis\(^6\) will prove this statement.

We next consider the possibility of extending these theorems to functions \( x(\tau) \) on \( \mathcal{A} \) to a Banach space. The theorem is not in general valid for additive bounded set functions of this type. This is illustrated by \( x(\tau) \) defined on all subsets of \( T = (0, 1) \) to the space \( X \) of bounded functions on \( S = (0, 1) \) where \( x(\tau) \) is the characteristic function of the subset of \( S \) which has the same coordinate values as \( \tau \). Clearly there exists no denumerable set \( R \) such that \( x(\tau - R\tau) = 0 \) for all denumerable sets \( \tau \).

However analogous theorems are obtained for c.a. set functions on \( \mathcal{A} \) to \( X \).

Theorem 3. Let $\aleph$ be an infinite cardinal number not greater than $\mathcal{T}$. For every $x \in C_x$ there exists a unique decomposition $x = x_1 + x_2$ and a set $R(x) \in 3$ of cardinal power not greater than $\aleph$ such that $x_1, x_2 \in C_x$, $x_1(\tau) = x(R \cdot \tau)$, $x_2(\tau) = 0$ if $\forall \tau \leq \aleph$. 

$x(\tau) \neq 0$ on at most a denumerable number of disjoint sets of 3. Suppose the contrary. Then there exists a denumerable sequence of disjoint sets $\{\tau_i\}$ and an $\varepsilon > 0$ such that $\|x(\tau_i)\| > \varepsilon$, $(i = 1, 2, \cdots )$. As $x(\tau)$ is c.a., $\sum_i x(\tau_i)$ converges. The supposition is therefore false.

The argument used in Theorem 1 will now prove the theorem.

Theorem 4. For every nonsingular $x \in C_x$, there exists an unique decomposition $x = \sum x_i$, the sum being unconditionally convergent, and such that $x_i$ is $\aleph_i$-homogeneous and $\aleph_i \neq \aleph_j$ if $i \neq j$.

The proof is identical with that of I in Theorem 2. Again there will exist disjoint $\overline{R}_\mu$'s such that $x_\mu(\tau) = x(\overline{R}_\mu \cdot \tau)$.

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