SOME PROBLEMS IN INTERPOLATION BY CHARACTERISTIC FUNCTIONS OF LINEAR DIFFERENTIAL SYSTEMS OF THE FOURTH ORDER

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In this paper we consider the convergence to \( f(x) \), defined on \([0, 1]\), of

\[
\Sigma_p[f(x)] = \alpha_0u_0(x) + \alpha_1u_1(x) + \cdots + \alpha_pu_p(x),
\]

where \( u_n(x) \), \((n = 0, 1, \cdots, p)\), are characteristic functions of certain self-adjoint linear differential systems of fourth order,

\[
\alpha_n = \sum_{k=0}^{p} f(x_k)u_n(x_k) \left\{ \sum_{k=0}^{p} u_n(x_k) \right\}^{-1}, \quad n = 0, 1, \cdots, p,
\]

and the symbol \( \sum' \) is used in the sense \( \sum_{k=0}^{p} y_k = y_0/2 + \sum_{k=1}^{p} y_k \). Throughout the discussion, \( x_k = 2k/(2p + 1) \), \((k = 0, 1, \cdots, p)\). The differential systems considered are

\[
u^{(iv)} - \rho u = 0,
\]

with boundary conditions

I. \( u'(0) = 0, u''(0) = 0, u'(1) = 0, u''(1) + u(1) = 0,\)

II. \( u'(0) = 0, u''(0) = 0, u'(1) + u(1) = 0, u''(1) + u'(1) = 0,\)

III. \( u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) + u'(1) = 0,\)

IV. \( u'(0) = 0, u''(0) = 0, u(1) = 0, u''(1) + u'(1) = 0,\)

V. \( u(0) = 0, u'(0) = 0, u(1) = 0, u'(1) = 0,\)

VI. \( u'(0) = 0, u''(0) = 0, u(1) = 0, u'(1) = 0.\)

The following theorems may be proved for these systems respectively.

I, II. If \( f(x) \) is continuous and of bounded variation in \([0, 1]\), then \( \lim_{p \to \infty} \Sigma_p[f(x)] = f(x) \) uniformly in \([0, 1]\).

III. If \( f(x) \) is continuous and of bounded variation in \([0, 1]\) and \( f(0) = f(1) = 0 \), then \( \lim_{p \to \infty} \Sigma_p[f(x)] = f(x) \) uniformly in \([0, 1]\).

IV. If \( f(x) \) is continuous and of bounded variation in \([0, 1]\) and \( f(1) = 0 \), then \( \lim_{p \to \infty} \Sigma_p[f(x)] = f(x) \) uniformly in \([0, 1 - \eta]\).

V, VI. If \( f(x) \) satisfies a Lipschitz condition in \([0, 1]\) and \( f(0) = f(1) = 0 \), then \( \lim_{p \to \infty} \Sigma_p[f(x)] = f(x) \) uniformly in \([\eta, 1 - \eta]\).

Here and hereafter \( \eta > 0 \) is arbitrarily small but fixed.

The method of proof for these theorems, as well as for those to fol-
low, is essentially the same as that of C. M. Jensen [1], who considered convergence properties of a somewhat similar sum using Sturm-Liouville functions.

The following equivalence theorems (the term “equivalence” is used in the sense of Jensen) may be proved for systems I; II, III, IV, respectively.

I, II. If \( f(x) \) is continuous in \([0, 1]\), then

\[
\lim_{p \to \infty} \left\{ \Sigma_p [f(x)] - T_p [f(x)] \right\} = 0
\]

uniformly in \([0, 1]\), where \( T_p [f(x)] \) is the cosine interpolation formula

\[
T_p [f(x)] = a_{0p} + a_{1p} \cos \pi x + \cdots + a_{pp} \cos p\pi x,
\]

\[
a_{np} = \sum_{k=0}^{p} f(x_k) \cos n\pi x_k \left\{ \sum_{k=0}^{p} \cos^2 n\pi x_k \right\}^{-1}
\]

\[
= \begin{cases} 
\frac{4}{2p + 1} \sum_{k=0}^{p} f(x_k) \cos n\pi x_k, & n = 1, 2, \ldots, p, \\
\frac{2}{2p + 1} \sum_{k=0}^{p} f(x_k), & n = 0.
\end{cases}
\]

III. If \( f(x) \) is continuous in \([0, 1]\), then

\[
\lim_{p \to \infty} \left\{ \Sigma_p [f(x)] - \bar{T}_p [f(x)] \right\} = 0
\]

uniformly in \([\eta, 1-\eta]\), where \( \bar{T}_p [f(x)] \) is the sine interpolation formula

\[
\bar{T}_p [f(x)] = \bar{a}_{1p} \sin \pi x + \bar{a}_{2p} \sin 2\pi x + \cdots + \bar{a}_{pp} \sin p\pi x,
\]

\[
\bar{a}_{np} = \sum_{k=0}^{p} f(x_k) \sin n\pi x_k \left\{ \sum_{k=0}^{p} \sin^2 n\pi x_k \right\}^{-1}
\]

\[
= \frac{4}{2p + 1} \sum_{k=0}^{p} f(x_k) \sin n\pi x_k, \quad n = 1, 2, \ldots, p.
\]

IV. If \( f(x) \) is continuous in \([0, 1]\), then

\[
\lim_{p \to \infty} \left\{ \Sigma_p [f(x)] - U_p [f(x)] \right\} = 0
\]

uniformly in \([\eta, 1-\eta]\), where \( U_p [f(x)] \) is given by

\[
U_p [f(x)] = b_{0p} \cos \left( \frac{\pi}{2} \right) x + b_{1p} \cos \left( \frac{3\pi}{2} \right) x + \cdots + b_{pp} \cos \left( p + \frac{1}{2} \right) \pi x,
\]

\[
b_{np} = \sum_{k=0}^{p} f(x_k) \cos (n + 1/2)\pi x_k \left\{ \sum_{k=0}^{p} \cos^2 (n + 1/2)\pi x_k \right\}^{-1}
\]
The first set of theorems also holds using characteristic functions of $u^{(v)} - [p^2 + \lambda(x)]u = 0$, with the above boundary conditions, $\lambda(x)$ being an arbitrary continuous function. Here, provided $\partial u(x) \neq 0$, the term $\int_0^1 u(x) \sum_{k=0}^p f(x_k) \{ \sum_{k=0}^p u(x_k) \}^{-1}$ is adjoined to $\Sigma_p [f(x)]$, where $\partial u(x)$ is the characteristic function corresponding to $\rho = 0$. (For $\lambda(x) = 0, \rho = 0$ is not a characteristic number.)

We give here the proof of the theorem for system V. System V is the problem of lateral vibrations of an elastic homogeneous rod clamped at both ends [2]. We first state two lemmas.

**Lemma 1.** If $f(x)$ is continuous and of bounded variation in $[0, 1]$ and $f(0) = f(1) = 0$, then $W_p[f(x)]$ and $w_p[f(x)]$ tend to the same limit $M(x)$ uniformly in $[\eta, 1 - \eta]$ as $p \to \infty$, where

$W_p[f(x)] = c_0[f(x)] + \frac{2}{2p + 1} \sum_{k=0}^p f(x_k) [\cos (n + 1/2)\pi x_k \; \sin (n + 1/2)\pi x_k]$, 

$w_p[f(x)] = c_0[f(x)] + \frac{2}{2p + 1} \sum_{k=0}^p f(x_k) [\cos (n + 1/2)\pi x_k \; \sin (n + 1/2)\pi x_k]$, 

$c_n = \int_0^1 f(t) [\cos (n + 1/2)\pi t \; \sin (n + 1/2)\pi t] dt$, 

$n = 0, 1, \ldots, p$.

**Lemma 2.** If $f(x)$ satisfies a Lipschitz condition in $[0, 1]$ and $f(0) = f(1) = 0$, then there exists a constant $C$, depending only on $\eta$, such that for $n$ sufficiently great, $\rho \geq n$, and $x$ in $[\eta, 1 - \eta]$, 

$| \alpha_n u_n(x) - c_n[f(x)] | < C/n^2$.

To prove Lemma 1, write

$W_p[f(x)] = 1W_p[f(x)] + 2W_p[f(x)] - 3W_p[f(x)]$, 

$w_p[f(x)] = 1w_p[f(x)] + 2w_p[f(x)] - 3w_p[f(x)]$. 

where

\[ 1W_p[f(x)] = \frac{2}{2p + 1} \sum_{k=0}^{p} f(x_k) \sum_{n=0}^{p} \cos (n + 1/2)\pi x_k \cos (n + 1/2)\pi x, \]

\[ 2W_p[f(x)] = \frac{2}{2p + 1} \sum_{k=0}^{p} f(x_k) \sum_{n=0}^{p} \sin (n + 1/2)\pi x_k \sin (n + 1/2)\pi x, \]

\[ 3W_p[f(x)] = \frac{2}{2p + 1} \sum_{k=0}^{p} f(x_k) \sum_{n=0}^{p} \sin (n + 1/2)\pi (x_k + x), \]

\[ 1w_p[f(x)] = \int_0^1 f(t) \sum_{n=0}^{p} \cos (n + 1/2)\pi t \cos (n + 1/2)\pi x dt, \]

\[ 2w_p[f(x)] = \int_0^1 f(t) \sum_{n=0}^{p} \sin (n + 1/2)\pi t \sin (n + 1/2)\pi x dt, \]

\[ 3w_p[f(x)] = \int_0^1 f(t) \sum_{n=0}^{p} \sin (n + 1/2)\pi (t + x) dt. \]

We first show that if \( f(x) \) is continuous and of bounded variation in \([0, 1]\), then \( \lim_{p \to \infty} 1W_p[f(x)] = (1/2)f(x) \) uniformly in \([0, 1] \). We employ the cosine interpolation formula \( T_p[f(x)] \). We shall use \( r_n(x) \) as generic notation for a function uniformly bounded in \( n \) and for \( x \) in \([0, 1]\), unless the range for \( x \) is otherwise stated, and \( r_n[r_n] \) for a quantity depending on \( n \) \([n, p]\) and uniformly bounded in \( n \) \([n, p]\). We have

\[ T_p[f(x)] = \frac{2}{2p + 1} \sum_{k=0}^{p} f(x_k) + \frac{4}{2p + 1} \sum_{k=0}^{p} f(x_k) \sum_{n=1}^{p} \cos n\pi x_k \cos n\pi x \]

\[ = \frac{1}{2p + 1} \sum_{k=0}^{p} f(x_k) \left[ \frac{\sin (p + 1/2)\pi (x_k - x)}{\sin (\pi/2)(x_k - x)} + \frac{\sin (p + 1/2)\pi (x_k + x)}{\sin (\pi/2)(x_k + x)} \right] \]

\[ = \frac{1}{2p + 1} \sum_{k=0}^{p} f(x_k) \left[ \frac{\sin p\pi (x_k - x) \cos (\pi/2)(x_k - x)}{\sin (\pi/2)(x_k - x)} + \frac{\sin p\pi (x_k + x) \cos (\pi/2)(x_k + x)}{\sin (\pi/2)(x_k + x)} \right] \]

\[ + \frac{2}{2p + 1} \cos p\pi x \sum_{k=0}^{p} f(x_k) \cos p\pi x_k. \]
Using auxiliary Lemma A stated below, we have

\[
T_p[f(x)] = \frac{1}{2p + 1} \sum_{k=0}^{p} f(x_k) \left[ \frac{\sin \frac{p\pi}{2} (x_k - x)}{\sin \frac{\pi}{2} (x_k - x)} + \frac{\sin \frac{p\pi}{2} (x_k + x)}{\sin \frac{\pi}{2} (x_k + x)} \right] + \frac{r_p(x)}{p},
\]

and

\[
1W_p[f(x)] = \frac{2}{2p + 1} \sum_{k=0}^{p} f(x_k) \sum_{n=0}^{p-1} \cos (n + 1/2)\pi x_k \cos (n + 1/2)\pi x + \frac{r_p(x)}{p}
\]

Thus

\[
1W_p[f(x)] - (1/2)T_p[f(x)] = \frac{1}{2(2p + 1)} \sum_{k=0}^{p} f(x_k) \left[ \frac{1 - \cos \frac{\pi}{2} (x_k - x)}{\sin \frac{\pi}{2} (x_k - x)} \sin \frac{p\pi}{2} (x_k - x) + \frac{1 - \cos \frac{\pi}{2} (x_k + x)}{\sin \frac{\pi}{2} (x_k + x)} \sin \frac{p\pi}{2} (x_k + x) \right] + \frac{r_p(x)}{p}
\]

By reason of the nature of Lemma A and the fact that for \( t \) in \( [0, 1] \) the functions \( \tan \frac{\pi}{4}(t-x) \) and \( \tan \frac{\pi}{4}(t+x) \) are of uniform bounded variation with respect to \( x \) in \( [0, 1-\eta] \), we have

\[
1W_p[f(x)] - (1/2)T_p[f(x)] = r_p(x)/p \quad \text{for} \quad x \in [0, 1-\eta].
\]

Hence \( [3] \)

\[
\lim_{p \to \infty} 1W_p[f(x)] = (1/2)f(x) \quad \text{uniformly in} \quad [0, 1-\eta].
\]

Employing \( t_p[f(x)] \), the partial sum of order \( p \) in the Fourier cosine series,

\[
t_p[f(x)] = a_0 + a_1 \cos \pi x + \cdots + a_p \cos p\pi x,
\]

\[
a_n = \frac{1}{\int_0^1 \cos^2 n\pi x dx} = \begin{cases} 
2 \int_0^1 f(x) \cos n\pi x dx, & n = 1, 2, \ldots, p, \\
\int_0^1 f(x) dx, & n = 0,
\end{cases}
\]
we may show similarly that if \( f(x) \) is continuous and of bounded variation in \([0, 1]\), then \( \lim_{p \to \infty} sW_p[f(x)] = (1/2)f(x) \) uniformly in \([0, 1-\eta]\). Employing \( T_p[f(x)] \), the sine interpolation formula, and \( l_p[f(x)] \), the partial sum of order \( p \) in the Fourier sine series,

\[
l_p[f(x)] = a_1 \sin \pi x + a_2 \sin 2\pi x + \cdots + a_p \sin p\pi x,
\]

\[
a_n = \frac{\int_0^1 f(x) \sin n\pi x \, dx}{\int_0^1 \sin^2 n\pi x \, dx}
= 2 \int_0^1 f(x) \sin n\pi x \, dx, \quad n = 1, 2, \ldots, p,
\]

we may likewise show that if \( f(x) \) is continuous and of bounded variation in \([0, 1]\) and \( f(0) = f(1) = 0 \), then

\[
\lim_{p \to \infty} sW_p[f(x)] = (1/2)f(x), \quad \lim_{p \to \infty} sW_p[f(x)] = (1/2)f(x)
\]

uniformly in \([0, 1-\eta]\). In connection with \( 1w_p[f(x)] \) and \( 2w_p[f(x)] \) we use Lemma B, stated below.

Finally, we show that if \( f(x) \) is of bounded variation in \([0, 1]\), then \( sW_p[f(x)] \) and \( sW_p[f(x)] \) both tend to \((1/2)\int_0^1 f(t) \{ \sin (\pi/2)(t+x) \}^{-1} \, dt\) uniformly in \([\eta, 1-\eta]\). We have

\[
sW_p[f(x)] = \frac{1}{2} \int_0^1 f(t) \frac{1 - \cos (p + 1)\pi(t + x)}{\sin (\pi/2)(t + x)} \, dt.
\]

For \( x \) in \([\eta, 1-\eta]\),

\[
sW_p[f(x)] = \frac{1}{2} \int_0^1 f(t) \frac{\cos (p + 1)\pi(t + x)}{\sin (\pi/2)(t + x)} \, dt
- \frac{1}{2} \int_0^1 f(t) \frac{\cos (p + 1)\pi(t + x)}{\sin (\pi/2)(t + x)} \, dt.
\]

Effecting some trigonometric reductions on the second integrand and using Lemma B, we have

\[
sW_p[f(x)] = \frac{1}{2} \int_0^1 f(t) \frac{\cos (p + 1)\pi(t + x)}{\sin (\pi/2)(t + x)} \, dt + \frac{r_p(x)}{p}.
\]

We may then prove that

\[
\lim_{p \to \infty} sW_p[f(x)] = \frac{1}{2} \int_0^1 f(t) \frac{1}{\sin (\pi/2)(t + x)} \, dt.
\]
uniformly in $[\eta, 1-\eta]$, using the fact just established that
\[
\lim_{p \to n} s_w[p(f(x))] = \frac{1}{2} \int_0^1 \frac{f(t)}{\sin(\pi/2)(t + x)} \, dt
\]
uniformly in $[\eta, 1-\eta]$, together with Lemma C, stated below.

**Lemma A.** If $f(x)$ is of bounded variation in $[0, 1]$, then for $n = 1, 2, \ldots, 2p$,
\[
\frac{1}{2^p + 1} \left| \sum_{k=0}^{p'} f(x_k) \cos n \pi x_k \right| = \frac{r_{np}}{n},
\]
\[
\frac{1}{2^p + 1} \left| \sum_{k=0}^{p'} f(x_k) \sin n \pi x_k \right| = \frac{r_{np}}{n}.
\]
Also
\[
\frac{1}{2^p + 1} \left| \sum_{k=0}^{p'} f(x_k) \sin (2^p + 1) \pi x_k \right| = 0.
\]

**Lemma B.** If $f(x)$ is of bounded variation in $[0, 1]$, then for $n > 0$,
\[
\left| \int_0^1 f(x) \cos n \pi x \, dx \right| = \frac{r_n}{n}, \quad \left| \int_0^1 f(x) \sin n \pi x \, dx \right| = \frac{r_n}{n}.
\]

**Lemma C.** If $f(x)$ is of bounded variation in $[0, 1]$, then for any pre-assigned $\varepsilon > 0$ there exists $Q$ such that for $p > q \geq Q$ and $x$ in $[\eta, 1]$,
\[
\frac{2}{2^p + 1} \left| \sum_{k=0}^{p'} f(x_k) \sum_{n=q+1}^{p} \sin (n + 1/2) \pi(x_k + x) \right| < \varepsilon.
\]

Lemma 2 is proved by means of auxiliary Lemmas D, E, and A.

**Lemma D.** For $n$ sufficiently great, and $p \geq n$,
\[
\left[ \sum_{k=0}^{p'} u_n^2(x_k) \right]^{-1} = \frac{2}{2^p + 1} \left( 1 + \frac{r_{np}}{n} \right).
\]

**Lemma E.** If $f(x)$ satisfies a Lipschitz condition in $[0, 1]$ and if $f(0) = f(1) = 0$, then for $n = 1, 2, \ldots, p$,
\[
\frac{1}{2^p + 1} \left| \sum_{k=0}^{p'} f(x_k) \exp \left\{ -(n + 1/2) \pi x_k \right\} \right| = \frac{r_{np}}{n^2},
\]
\[
\frac{1}{2^p + 1} \left| \sum_{k=0}^{p'} f(x_k) \exp \left\{ -(n + 1/2) \pi(1 - x_k) \right\} \right| = \frac{r_{np}}{n^2}.
\]
The following asymptotic expression is known \[4\] for the characteristic functions of system V:

\[
u_n(x) = \cos (n + 1/2)\pi x - \sin (n + 1/2)\pi x
+ (-1)^n \exp \{-(n + 1/2)\pi(1 - x)\}
- \exp \{-(n + 1/2)\pi x\} + \exp \{-(n + 1/2)\pi\} r_n(x).
\]

Define

\[
\sigma_p[f(x)] = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \cdots + \alpha_p u_p(x),
\]

\[
\alpha_n = \frac{\int_0^1 f(x) u_n(x) dx}{\int_0^1 u^2_n(x) dx}, \quad n = 0, 1, \cdots, p.
\]

Denote by \(\Sigma_p^{r,s}\) the sum of the terms in \(\Sigma_p[f(x)]\) with subscripts \(r\) to \(s\) inclusive; similarly in the other sums.

For \(x\) in \([\eta, 1-\eta]\),

\[
|f(x) - \Sigma_p[f(x)]| \leq |\sigma^{(0,N)} - \Sigma_p^{(0,N)}| + |W_p^{(N+1,p)} - \Sigma_p^{(N+1,p)}|
+ |W_p^{(0,N)} - M(x) - W_p^{(0,p)}|
+ |M(x)| + |f(x) - \sigma^{(0,N)}|.
\]

The right-hand member can be made arbitrarily small by choosing \(p\) sufficiently large. Call the six terms \(D_1, D_2, \cdots, D_6\). Given \(\varepsilon\), first choose \(N\) sufficiently large so that \(D_2 < \varepsilon, D_3 < \varepsilon, D_4 < \varepsilon\) for \(p \geq N+1\). Having fixed \(N\), choose \(P\) sufficiently large so that \(D_1 < \varepsilon, D_3 < \varepsilon, D_4 < \varepsilon\) for \(p \geq P\). It remains to justify these statements. For \(D_6\) we use a result in a paper by J. D. Tamarkin \[5\]. For \(D_5\) we use Lemma 2, and for \(D_4\) and \(D_5\) Lemma 1. In \(D_1\) and \(D_2\) we are dealing essentially with integrals of continuous functions and the sums which tend to the integrals as limits, the number of terms being finite.

Now consider \(u^{(iv)} - \rho^4 u = 0, \lambda(x)\) being an arbitrary continuous function. A fundamental system of solutions of \(u^{(iv)} - \rho^4 u = 0\) is \(u_1 = \cos \rho x, u_2 = \sin \rho x, u_3 = e^{\rho x}, u_4 = e^{-\rho x}\). By the method of variation of constants we have, as an equation satisfied by \(u\),

\[
u = A \cos \rho x + B \sin \rho x + C e^{\rho x} + D e^{-\rho x}
+ \frac{1}{2\rho^4} \int_0^x \lambda(t) u(t) \left[- \sin \rho(x - t) + \frac{e^{\rho(x-t)}}{2} - \frac{e^{-\rho(x-t)}}{2}\right] dt,
\]
where $A, B, C, D$ are arbitrary constants. Applying the boundary conditions of system V and choosing the multiplicative constant so that $A = 1$, we have, assuming $\rho \neq 0$,

$$u = \cos \rho x - \sin \rho x$$

$$+ \left[ \sin \rho x \left\{ - 2e^{-\rho} \sin \rho + \frac{1}{2\rho^3} \int_0^1 \lambda(t)u(t) \left[ - e^{-\rho} \sin \rho(1 - t) - e^{-\rho} \cos \rho(1 - t) + e^{-\rho t} \right] dt \right\} + e^{-\rho(1-x)} \sin \rho - \frac{1}{4\rho^3} \int_0^1 \lambda(t)u(t) \left[ - e^{-\rho(1-x)} \sin \rho(1 - t) - e^{-\rho(1-x)} \cos \rho(1 - t) + e^{\rho(x-t)} \right] dt \right.$$ 

$$- e^{-\rho x} + e^{-\rho(1+x)} \cos \rho + \frac{1}{4\rho^3} \int_0^1 \lambda(t)u(t) \left[ - e^{-\rho(1+x)} \sin \rho(1 - t) - e^{-\rho(1+x)} \cos \rho(1 - t) + e^{\rho(x-t)} \right] dt \right] \left[ 1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho \right]^{-1}$$

$$+ \frac{1}{2\rho^3} \int_0^x \lambda(t)u(t) \left[ - \sin \rho(x - t) + \frac{e^{\rho(x-t)}}{2} - \frac{e^{-\rho(x-t)}}{2} \right] dt,$$

and, as an equation satisfied by the characteristic numbers $\rho = \rho_n$,

$$\cos \rho = 2e^{-\rho} - e^{-2\rho} \cos \rho$$

$$+ \frac{\sin \rho}{2\rho^3} \int_0^1 \lambda(t)u(t) \left[ e^{-\rho} \cos \rho(1 - t) - e^{-\rho t} \right] dt$$

$$+ \frac{\cos \rho}{2\rho^3} \int_0^1 \lambda(t)u(t) \left[ - e^{-\rho} \sin \rho(1 - t) + \frac{e^{-\rho t}}{2} - e^{-\rho(2-t)} \right] dt$$

$$+ \frac{1}{4\rho^3} \int_0^1 \lambda(t)u(t) \left[ \sin \rho(1 - t) - \cos \rho (1 - t) + e^{-\rho(1-t)} \right] dt$$

$$- e^{-\rho} \int_0^1 \lambda(t)u(t) \left[ - e^{-\rho} \sin \rho(1 - t) - e^{-\rho} \cos \rho(1 - t) + e^{-\rho t} \right] dt.$$

For each $\rho = \rho_n$, $u$ is a continuous function; we show $u$ uniformly bounded in $n$. In $u$, the terms

$$- \frac{1}{4\rho^3} \int_0^1 \lambda(t)u(t)e^{\rho(x-t)} dt \left[ 1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho \right]^{-1},$$

$$+ \frac{1}{4\rho^3} \int_0^x \lambda(t)u(t)e^{\rho(x-t)} dt$$

may be combined to give
\[ \left\{ - \frac{1}{4p^3} \int_{x}^{1} \lambda(t) u(t) e^{\rho x - \rho t} \, dl - \frac{\sin \rho + \cos \rho}{4p^3} \int_{0}^{x} \lambda(t) u(t) e^{-\rho (x-t)} \, dl \right\} \cdot \left\{ 1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho \right\}^{-1}. \]

For \( t \in [x, 1], x-t \leq 0 \); for \( t \in [0, x], 1-x+t \geq 0 \). Also, there exists a constant \( c > 0 \) such that \( 1-e^{-\rho} \sin \rho - e^{-\rho} \cos \rho \geq c \) for \( \rho = \rho_n \). Call \( M_n = \max_{[0,1]} |u_n(x)|, K = \int_{0}^{1} |\lambda(t)| \, dt \). Then

\[ |u_n(x)| \leq 2 + \frac{5}{c} + \frac{17M_nK}{4c\rho_n^3}, \quad M_n \leq 2 + \frac{5}{c} + \frac{17M_nK}{4c\rho_n^3}, \]

\[ M_n \leq \frac{2 + 5/c}{1 - 17K/(4c\rho_n^3)}. \]

Thus for \( n \) sufficiently great, \( M_n \leq 4 + 10/c \). The remaining \( n \)'s form a finite set. Hence \( u_n(x) \) is uniformly bounded in \( n \). Thus

\[ u_n(x) = \cos \rho_n x - \sin \rho_n x + e^{-\rho_n (1-x)} \sin \rho_n - e^{-\rho_n x} + r_n(x)/\rho_n^3, \]

and \( \cos \rho_n = \phi(\rho_n) \), where \( \lim_{n \to \infty} \phi(\rho_n) = 0 \). Thus \( \rho_n = (n+1/2)\pi + \epsilon_n \), where \( \lim_{n \to \infty} \epsilon_n = 0 \). It follows that

\[ u_n(x) = \cos (n + 1/2)\pi x - \sin (n + 1/2)\pi x \]

\[ + (-1)^n \exp \left\{ - (n + 1/2)\pi (1 - x) \right\} \]

\[ - \exp \left\{ - (n + 1/2)\pi x \right\} + r_n(x)/n^3. \]

The theorem for system V follows.

**References**


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