AN ALMOST UNIVERSAL FORM

GORDON PALL

P. R. Halmos\textsuperscript{1} obtained the 88 possible forms \((a, b, c, d)\), \(0 < a \leq b \leq c \leq d\), which represent all positive integers with one exception, and proved that property for all except for the form \(h = (1, 2, 7, 13)\). A proof for \(h\) follows.

The forms \(f = (1, 2, 7)\) and \(g = (1, 1, 14)\) constitute the reduced forms of a genus.\textsuperscript{2} Between them they represent all positive integers not of the form \(\Lambda = 7^{2k+1}(7m+3, 5, 6)\). The identities

\[
x^2 + y^2 + 14z^2 = x^2 + 2((y + 7z)/3)^2 + 7((y - 2z)/3)^2 \\
y^2 + 2((x + 7z)/3)^2 + 7((x - 2z)/3)^2
\]

show that every number represented by \(g\) with either \(y \equiv -z\) or \(x \equiv -z \pmod{3}\) is also represented by \(f\). Hence every number \(3n\) and \(3n+1\) not of the form \(\Lambda\) is represented by \(f\). For, \(x = y = 0, z \neq 0\), and \(x, y \neq 0, z = 0 \pmod{3}\) both imply \(g \equiv 2\). If \(N = 3n\) or \(3n + 1\) is of the form \(\Lambda\), then \(7 | N\), so that \(N - 13 \cdot 3^2 \neq \Lambda\). Similarly, one of \(3n + 2 - 13\) and \(3n + 2 - 52\) is not of the form \(\Lambda\); but neither of these is congruent to \(2 \pmod{3}\). These linear forms are positive if \(n \geq 39\); \(h\) represents all integers not less than 119. The only number less than 119 not represented in \((1, 2, 7, 13)\) is found to be 5.

\textsuperscript{1} This Bulletin, vol. 44 (1938), pp. 141-144.
\textsuperscript{2} See any table of positive ternaries.
\textsuperscript{3} For example, see B. W. Jones, Transactions of this Society, vol. 33 (1931), pp. 111-124.