By Theorem 2, the solutions of the equation (17) are given by (16).

If \( x_i = \rho_i, y_k = \sigma_k \) is any solution of (13) and we choose \( \alpha_i = \rho_i, \mu_k = \sigma_k, \lambda = f(\rho) \), we have that \( s = 0 \) and the solution becomes \( x_i = \rho_i K^{n-1}, y_k = \sigma_k K^{n+1} \), where \( K = A\lambda(AD-BC) \), which is equivalent to the given solution provided \( K \neq 0 \); that is, provided \( x_i = \rho_i, y_k = \sigma_k \) is not a solution of (14). It will be noted that if \( K \neq 0 \), then \( t \neq 0 \).

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A MULTIPLE NULL-CORRESPONDENCE AND A SPACE CREMONA INVOLUTION OF ORDER \( 2n-1 \)

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PART I. A NULL-SYSTEM \((1, mn, m+n)\) BETWEEN THE PLANES
AND POINTS OF SPACE \((m, n=1, 2, 3, \ldots)\)

1. Introduction. Consider a curve \( \delta_m \) of order \( m \) having \( m-1 \) points in common with a straight line \( d \), and a curve \( \delta_n' \) of order \( n \) having \( n-1 \) points in common with a straight line \( d' \), \((m, n=1, 2, 3, \ldots)\).

It is assumed for the present that neither \( \delta_m \) nor \( d \) intersects either \( \delta_n' \) or \( d' \).

In general, through any point \( P \) of space there passes one ray \( \rho \) which intersects \( \delta_m \) once and \( d \) once, and one ray \( \rho' \) which intersects \( \delta_n' \) once and \( d' \) once; \( \rho \) and \( \rho' \) determine a plane \( \pi \), the null-plane of \( P \).

Conversely, a plane \( \pi \) determines \( m \) rays \( \rho_i \) and \( n \) rays \( \rho_i' \) lying in it which intersect, a ray \( \rho \) with a ray \( \rho' \), in \( mn \) points, the null-points of the plane \( \pi \).

Any point \( \alpha \) in general position determines a ray \( \rho \). As \( \alpha \) describes a line \( l \), the plane \( \pi \) of \( \rho \) and \( l \) contains \( n \) rays \( \rho_i' \), which intersect \( l \) in \( n \) points \( \beta \); conversely, any point \( \beta \) on \( l \) determines a ray \( \rho' \) which determines with \( l \) the plane \( \pi \), and \( \pi \) contains \( m \) rays \( \rho \) which intersect \( l \) in \( m \) points \( \alpha \)—one being the original \( \alpha \). Thus an \((m, n)\) correspondence is set up among the points of \( l \) with valence zero; there are \( m+n \) coincidences and therefore \( m+n \) points on any line \( l \) whose null-planes contain \( l \).

2. Planes whose null-points behave peculiarly. We can obtain the last result by another method; this will yield additional information about planes whose null-points behave peculiarly.

Let a plane \( \pi \) turn about a line \( l \) as axis. A ruled surface will be generated by the \( m \) rays \( \rho_i \) lying in \( \pi \). This surface is of order \( m+1 \); \( \delta_m \) is a onefold curve on the surface and \( d \) is an \( m \)-fold line. Another

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1 Presented to the Society, December 2, 1939.
ruled surface will be generated in this manner by the rays $p'$ lying in \( \pi \); its order is \( n+1 \), \( \delta' \) is a onefold curve and \( d' \) is an \( n \)-fold line on this surface. The curve of intersection of these two surfaces is of order \((m+1)(n+1)\) and consists of \( l \) and a twisted curve \( k_{mn+m+n} \) of order \((m+1)(n+1)-1=mn+m+n\). This \( k_{mn+m+n} \) is the locus of the null-points of all planes \( \pi \) through \( l \).

Since a plane \( \pi \) meets this in \( mn \) points outside \( l \), \( k_{mn+m+n} \) must intersect \( l \) in \( m+n \) points through each of which a ray \( \rho \) and a ray \( \rho' \) pass which are coplanar with \( l \). Call such a point on \( l \), \( P \). The plane \( \rho \rho' \) is the null-plane of \( P \) and has \( mn-1 \) null-points outside \( l \), and it follows that plane \( \rho \rho' \) is tangent to \( k_{mn+m+n} \) at \( P \). The null-planes of the \( m+n \) points of intersection of \( k_{mn+m+n} \) with \( l \) are tangent planes of \( k_{mn+m+n} \) at these points.

The line \( d \), an \( m \)-fold line on the first of the two surfaces described above, intersects the second surface in \( n+1 \) points, which are \( m \)-fold points on the first surface. The line \( d' \) intersects the first of the two surfaces in \( m+1 \) points which are \( n \)-fold points on the second surface. These points all lie on \( k_{mn+m+n} \) and the \( m+1 \) are \( n \)-fold points of \( k_{mn+m+n} \) and \( n+1 \) are \( m \)-fold points of \( k_{mn+m+n} \) has \( m+1 \) \( n \)-fold points on \( d' \) and \( n+1 \) \( m \)-fold points on \( d \).

\( \delta_m \) has no actual double points or other multiple points. It is, however, rational and has \( (m-1)(m-2)/2 \) apparent double points and its rank is \( r=m(m-1)-(m-1)(m-2)=2(m-1) \); that is, the order of its developable surface is \( 2(m-1) \). Similarly, the order of the developable surface of \( \delta'_n \) is \( 2(n-1) \). The line \( l \) will intersect \( 2(m-1) \) tangents of \( \delta_m \) and \( 2(n-1) \) tangents of \( \delta'_n \). In the plane \( \pi \) through \( l \) and a tangent line \( t \) of the first group, two rays \( \rho \) coincide in the line which joins the point of tangency of \( t \) with the intersection of \( d \) and \( \pi \). Of the \( mn \) null-points in the plane \( \pi \), \( n \) lie on each of the other \( m-2 \) rays \( \rho \), and \( 2n \) fall two and two together on the coinciding rays; in these points \( k_{mn+m+n} \) is tangent to the plane of \( l \) and \( t \) and the number of these planes is \( 2(m+n-2) \).

From the discussion of this section we have the following conclusions:

(1) The planes, \( m \) of whose null points coincide with a point of \( d \), envelope a surface of class \( n+1 \); and the planes, \( n \) of whose null points coincide with a point of \( d \), envelope a surface of class \( m+1 \).

(2) The planes, \( 2n \) of whose null-points coincide two and two on a ray \( \rho \), envelope a surface of class \( 2(m-1) \), \( n \) of the remaining null-points lying on each of the other \( m-2 \) rays \( \rho \); the planes, \( 2m \) of whose null-points coincide two and two on a ray \( \rho' \), envelope a surface of class
2(n−1), m of the remaining null-points lying on each of the other n−2 rays ρ′.

Consider a plane π through l, whose intersection with d is also an intersection with δ_m. Call this common point of d and δ_m, Δ. Then the rays ρ_i lying in π will be the m−1 lines joining Δ to the m−1 points of intersection of δ_m and π, not lying on d, and the line λ joining Δ to the intersection of l and the plane of d and the tangent line to δ_m at Δ. This line λ will be the limiting position of a ray ρ as a plane revolves about l into the position of π.

In the osculating planes of δ_m and δ'_m, three rays coincide. Therefore, in the osculating planes of δ_m, 3n of the null-points coincide three and three on the triple ray; in the osculating planes of δ'_m, 3m of the null-points coincide three and three on the triple ray.

3. Points whose null-planes behave peculiarly. Consider a point P on d. The point P determines one ρ'. Any plane π through ρ' determines m rays ρ through P. Therefore π counts m times as null-plane of P. Conversely, for every plane through ρ' there fall m null-points together at P. The surface of class n+1 mentioned in §2 must have the planes π as tangent planes. This surface is a ruled surface consisting of rays ρ' which intersect d, and conversely. Call this surface Σ.

The surface formed by rays ρ' which intersect a general straight line l is (§2) of order n+1, and d intersects this surface in n+1 points. Thus there are n rays ρ' which intersect d and also an arbitrary line l. Therefore the surface Σ is of degree n+1. The line d is a onefold directrix on Σ_{n+1} and d′ is an n-fold directrix; for, the n-ic cone of δ'_m projected from a point of d′ will intersect d in n points. The locus of points whose null-planes have m null-points coinciding is Σ_{n+1}.

Similarly, the ruled surface Σ_{m+1} of order m+1, consisting of rays ρ that intersect d′, is the locus of points whose null-planes have n null-points coinciding.

Now Σ_{n+1} and Σ_{m+1} have mn+1 generators in common. For the congruence of rays ρ has the characteristic (1, m) and the congruence of rays ρ′ has the characteristic (1, n) so that, from Halphen’s theorem,² there are 1·1+m·n=mn+1 common rays.

Since both rays ρ and ρ’ through any point on one of these mn+1 common rays coincide, any plane through the ray can be taken as null-plane of the point. Every plane of the pencil through any one of the mn+1 common rays has m null-points coinciding on d and n null-points coinciding on d′.

² C. M. Jessop, A Treatise on the Line Complex, 1903, p. 259.
The intersection of $\Sigma_{n+1}$ and $\Sigma_{m+1}'$ is of degree $(n+1)(m+1)$. Since $d'$ was shown to be an $n$-fold line on $\Sigma_{n+1}$ and is clearly a onefold line on $\Sigma_{m+1}'$, $d'$ therefore counts $n$ times in the intersection of these two surfaces. Similarly $d$ counts $m$ times in the intersection. Each of the $mn+1$ common rays of the two congruences counts once in the intersection. The parts just enumerated have total degree $n+m+mn+1 = (n+1)(m+1)$. Therefore, the locus of points whose null-planes have $m$ null-points coinciding in one point and $n$ null-points coinciding in another consists of the lines $d$ and $d'$ and the $mn+1$ common rays of the two congruences.

Now consider a plane containing $d$; let it intersect $d'$ in $D'$ and $\delta_n'$ in $n$ points $N_i$. Every point of the $n$ lines $D'N_i$ is a null-point of this plane—similarly for planes through $d'$.

Let point $P$ be on $\delta_m$ but not on $d$. One $\rho'$ is determined but every line from $P$ to $d$ will be a $\rho$. Therefore, any point of $\delta_m$ or $\delta_n'$ not also a point of $d$ or $d'$ has the pencil of planes through the ray of the opposite congruence as null-planes.

**PART II. A SPACE CREMONA INVOLUTION OF ORDER $2n-1$ ($n$ ANY INTEGER)**

4. **Definition.** Not every skew curve of order $n$ has a secant meeting it in $n-1$ points, and some have only one such secant, but there are also skew curves of order $n$ that have two $(n-1)$-secant lines. In such case they lie on a quadric surface and have a singly infinite system of such secants. The two selected must be two generators of the same regulus.

Consider a fixed twisted curve $\delta_n$ of order $n$ having $n-1$ points in common with a fixed line $d$ and $n-1$ points in common with another fixed line $d'$. This construction occurs when the two twisted curves $\delta_n'$ and $\delta_m$ in Part I are identical but lines $d$ and $d'$ remain skew to each other.

A general point $P$ determines a unique line intersecting $\delta_n$ once, at $A$, and $d$ once, at $D$, and a unique line intersecting $\delta_n$ once, at $B$, and $d'$ once, at $D'$. We define $P'$, the correspondent of $P$, to be the intersection of lines $AD'$ and $BD$. It is an involution.

5. **Equations.** Let $d$ be $x_1 = 0$, $x_2 = 0$, and $d'$ be $x_3 = 0$, $x_4 = 0$, and the parametric equations of $\delta_n$ be

$$x_1 = (as + bt) \prod_{1}^{n-1} (t_is - s_it), \quad x_2 = (cs + dt) \prod_{1}^{n-1} (t_is - s_it),$$

$$x_3 = (es + ft) \prod_{n}^{2n-2} (t_is - s_it), \quad x_4 = (gs + ht) \prod_{n}^{2n-2} (t_is - s_it),$$
where \((s_i, t_i), (i = 1, 2, \cdots, n - 1)\), are values of the parameter at the \(n - 1\) points of \(\delta_n\) on \(d\), and for \(i = n, n + 1, \cdots, 2n - 2\) are values of the parameter at the \(n - 1\) points of \(\delta_n\) on \(d'\). Then the equations of the involution are

\[
x'_1 = (ad - bc) \left[ (ah - bg)x_3 - (af - be)x_4 \right] \prod_{1}^{n-1} \alpha_i \prod_{1}^{n-1} \beta_i,
\]

\[
x'_2 = (ad - bc) \left[ (ch - dg)x_3 - (cf - de)x_4 \right] \prod_{1}^{n-1} \alpha_i \prod_{1}^{n-1} \beta_i,
\]

\[
x'_3 = (fg - eh) \left[ (cf - de)x_1 - (af - be)x_2 \right] \prod_{1}^{2n-2} \alpha_i \prod_{1}^{2n-2} \beta_i,
\]

\[
x'_4 = (fg - eh) \left[ (ch - dg)x_1 - (ah - bg)x_2 \right] \prod_{1}^{2n-2} \alpha_i \prod_{1}^{2n-2} \beta_i,
\]

where \(\alpha_i = (t_i d + s_i c)x_1 - (t_i b + s_i a)x_2\) and \(\beta_i = (t_i h + s_i g)x_3 - (t_i b + s_i e)x_4\).

It is of order \(2n - 1\), \(n\) any integer.

6. The fundamental system. Line \(d\) is an \((n - 1)\)-fold fundamental line of simple contact. The \(n - 1\) fixed tangent planes through \(d\) are \(\alpha_i = 0\), \((i = 1, 2, \cdots, n - 1)\). The line \(d\) is an \(F\)-line of the first species whose principal surface consists in the \(n - 1\) planes \(\beta_i = 0\), \((i = 1, 2, \cdots, n - 1)\).

Line \(d'\) is an \((n - 1)\)-fold \(F\)-line of simple contact. The \(n - 1\) fixed tangent planes through \(d'\) are \(\beta_i = 0\), \((i = n, n + 1, \cdots, 2n - 2)\). \(d'\) is an \(F\)-line of the first species whose \(P\)-surface is \(\prod_{1}^{2n-2} \alpha_i = 0\).

Points \(\Delta_i\), \((i = 1, 2, \cdots, n - 1)\), intersections of \(d\) with \(\delta_n\) whose parameters on \(\delta_n\) are \((s_i, t_i)\), and points \(\Delta'_i\), \((i = n, n + 1, \cdots, 2n - 2)\), intersections of \(d'\) with \(\delta_n\), are isolated \(n\)-fold \(F\)-points whose \(P\)-surfaces are, respectively, the above mentioned fixed tangent planes \(\alpha_i = 0\), \((i = 1, 2, \cdots, n - 1)\), and \(\beta_i = 0\), \((i = n, n + 1, \cdots, 2n - 2)\).

The \((n - 1)^2\) lines, each joining a \(\Delta_i\) to a \(\Delta'_i\), are simple \(F\)-lines without contact. They are \(F\)-lines of the second species.

The \((n - 1)^2\) lines of intersection of the fixed tangent planes through \(d\) with the fixed tangent planes through \(d'\) are simple \(F\)-lines without contact. They are \(F\)-lines of the second species.

7. Invariant locus. Every point of the curve \(\delta_n\) is invariant. Every line that intersects \(d\), \(d'\), and \(\delta_n\), each once, goes over into itself although it is not pointwise invariant. The locus of these lines is the quadric surface on which \(d\), \(d'\), and \(\delta_n\) lie.

8. Intersection of two homaloids. Since they are surfaces of order
$2n - 1$, two homaloids intersect in a space curve of order $(2n - 1)^2$.

The fixed part of this curve consists in the lines $d$ and $d'$, each counting $n(n - 1)$ times, the $(n - 1)^2$ lines joining the isolated $n$-fold $F$-points of $d$ with those of $d'$, each counting once, and the $(n - 1)^2$ lines of intersection of the fixed tangent planes through $d$ with those through $d'$, each counting once. The order of this fixed part is $2n(n - 1) + 2(n - 1)^2$.

The variable part of the curve of intersection is of order $2n - 1$ and corresponds to the line of intersection of the two general planes which go over into the pair of homaloids.

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