To separate real and imaginary parts we may write
\[(f_1^* + if_1)(f_2^* + if_2) = f_1^* f_2^* - f_1 f_2 + i \frac{d}{dx}(f_1 f_2^*).\]

Differentiating this last expression \(p\) times and taking absolute magnitudes, we obtain
\[
\left\{ \frac{d^{p+1}}{dx^{p+1}}(f_1 f_2^*) \right\}^2 + \left\{ \frac{d^p}{dx^p}(f_1^* f_2^* - f_1 f_2) \right\}^2 \leq (2^p)^2,
\]
which is more than we set out to prove. The functions \(f_1(x) = \sin (x + \alpha_1)\) and \(f_2(x) = \sin (x + \alpha_2)\) show that our constant is the "best possible."

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ON THE CARATHÉODORY CONDITION FOR UNILATERAL VARIATIONS

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The two formulations and proofs of the Carathéodory condition in the calculus of variations given by Graves do not necessarily apply to the case when the minimizing curve may have arcs in common with the boundary of the region of admissible variations. The purpose of this note is to show how his first formulation and proof can be modified so as to be applicable to unilateral (one-sided) variations in the plane.

An admissible curve
\[E_0: \quad x^a = x^a(t), \quad t_0 \leq t \leq T, \quad \alpha = 1, 2, \ldots, n,\]
which minimizes the integral
\[J = \int_{t_0}^{T} F(x_1, \ldots, x_n, x_1', \ldots, x_n') dt \equiv \int_{t_0}^{T} F(x, x') dt\]
in the class of all admissible curves joining two fixed points \(x_0\) and \(X\) in space of \(n\) dimensions (\(n > 1\)), must satisfy certain well known con-

1 Presented to the Society, December 29, 1939.
3 Cf. Mancill, The minimum of a definite integral with respect to unilateral variations, Contributions to the Calculus of Variations, 1933–37, University of Chicago, 1937, p. 121, condition \(C_{\infty}^0\).
conditions. Suppose that $E_0$ is an extremaloid joining the points $x_0$ and $X$ and having corners at $x_1, \ldots, x_m$, and suppose that the function $F_i$ is different from zero along $E_0$ including both sides of corners. If the function

$$\Omega_0(x, p^-, p^+) = (\partial F^+/\partial x^a)p^a^- - (\partial F^-/\partial x^a)p^a^+, \tag{1}$$

where $p^a^-$ and $p^a^+$ represent the direction cosines of the tangents to $E_0$ at the corners, is different from zero at each corner of $E_0$, there exists an $(n-1)$-parameter family

$$x^a = \phi^a(t, a) \tag{2}$$

of extremaloids defined for $t_0 - \delta \leq t \leq T + \delta$, passing through the point $x_0$ for $t = t_0$, and containing $E_0$ for $a^i = a^i_0$, $t_0 \leq t \leq T$. We shall now prove the following form of the Carathéodory condition:

**Theorem.** Let $E_0$ be an extremaloid joining the points $x_0$ and $X$ and minimizing the integral $J$, satisfying the following conditions:

1. $E_0$ is positively strong;\(^4\)
2. $\Omega_0 \neq 0$ at the corners on $E_0$;
3. $E_0$ is uniquely determined by each of its elements $(x, x')$;\(^7\)
4. the determinant

$$D(t, a_0) = \left| \begin{array}{cc} \frac{\partial \phi^a}{\partial a^i} & \frac{\partial \phi^a}{\partial t} \\ \end{array} \right|_{a=a_0} \tag{3}$$

does not vanish at the corners on $E_0$.

Then the determinant $D(t, a_0)$ does not change sign at the corners on $E_0$.

Suppose that the determinant $D(t, a_0)$ changes sign at a corner $x_k$ where $t = t_k$. Also, suppose that the corners on all extremaloids of the family occur for fixed values of the parameter $t$. Consider the one-parameter family of curves $E_u$ constructed by Graves.\(^9\) The value of the integral $J$ taken along $E_u$ is a function $J(u)$ whose derivative near $u = t_k$ is

$$J'(u) = - \mathcal{E}(\phi^a^-, \phi'^a^-, x'^a^+) \equiv - \mathcal{E}[\tilde{I}(u), \tilde{a}(u), x'^a(u)]. \tag{4}$$

The second derivative of $J(u)$ at $u = t_k$ is

For a full statement of the problem and the classical necessary conditions, see Graves, loc. cit., pp. 2–3.

\(^4\) Graves, loc. cit., p. 6.

\(^5\) Cf. Graves, loc. cit., p. 9.

\(^7\) Ibid., p. 11.

\(^8\) Ibid., p. 7.

on account of the extremal property of $E_0$, and since the derivative of the function $\mathcal{E}[t_k, \bar{a}(u), x^{\alpha+}(u)]$ is zero at $u = t_k$. This last statement follows from assumptions (1) and (3) of the theorem and the fact that we have assumed that the corners on all the extremaloids of the family occur for fixed values of the parameter $t$, and therefore the function $\mathcal{E}[t_k, \bar{a}(u), x^{\alpha+}(u)] \geq 0$ near $u = t_k$ and equals zero at $t_k$. By making use of the homogeneity property of the function $F$ and the expression for $\mathcal{I}'(t_k)$ given by Graves, the derivative (3) may be reduced to

\[
J''(t_k) = - \left[ (x^{\alpha-} - x^{\alpha-}) (x^{\alpha+} - x^{\alpha+}) \right]^{1/2}
\]

where $p^{\alpha-} = x^{\alpha-} / (x^{\alpha-} - x^{\alpha-})^{1/2}$ and $p^{\alpha+} = x^{\alpha+} / (x^{\alpha+} + x^{\alpha+})^{1/2}$. Consequently, $J(t_k)$ is a maximum of the function $J(u)$ for $u$ in a neighborhood of $t_k$, since $J'(t_k) = 0$ and $J''(t_k) < 0$. Thus, $E_0$ could not minimize the integral $J$.

Let us now consider the problem of unilateral variations in the plane. In this case it can be shown that a Carathéodory condition as stated here applies to every arc of the minimizing curve which is an arc of an extremaloid. The condition may be stated in terms of the one-parameter family of extremaloids through any fixed point of such an arc. For such arcs of the minimizing curve as $E_{23}$ immediately following a non-extremal arc $E_{12}$ of the boundary of the region of admissible curves whose direction at the point 2 where it meets $E_{12}$ is not directed towards the exterior side of $E_{12}$ but is as shown in Fig. 1, the Carathéodory condition may be stated in terms of the one-parameter family of extremaloids whose members, at least those members on the admissible side of the minimizing curve, are tangent to the arc $E_{12}$ of the boundary. In order to insure the existence of the families of extremaloids just described, it is necessary to assume that the continuity and homogeneity properties of the integrand $F$ hold in an extended region containing in its interior those arcs of the minimizing curve which are extremaloids in common with the boundary. It should also be pointed out that in the construction of the family of extremaloids tangent to $E_{12}$, it is necessary to assume that the Weierstrass $\mathcal{E}$-function is greater than or equal to zero along

\[^{10}\text{Loc. cit., p. 15.}\]

\[^{11}\text{Mancill, loc. cit., pp. 105–107. For a discussion of the case when the extremaloid is directed towards the exterior side of $E_{12}$ see pages 138–141. In this case the minimizing curve need not satisfy the corner conditions at the point 2.}\]
$E_{12}$ preceding 2. The family of extremaloids is then composed of the family of extremals tangent to $E_{12}$ in a neighborhood of and preceding the point 2, and the family of extremaloids containing the extremaloid $E_{23}$ which is the continuation family of the extremals tangent to $E_{12}$ near the point 2.

In order to show how the proof of the theorem applies in the way just described, let (1) represent the one-parameter family of extremaloids through a fixed point of such an arc of the boundary. Let $x_h$ be the first corner beyond this fixed point. Then the family of curves $E_u$ used in the proof of the theorem exists for $u$ near $t_h$ and its members are admissible curves for $u$ on one side of $t_h$. Therefore, the relation (2) holds here for $u$ on the admissible side of $t_h$. The function $-E[f(u), a(u), x'(u)]$ is defined locally and in a neighborhood of $t_h$. Its derivative at $u = t_h$ is $J''(t_h)$ as given in (4). Consequently, it follows from (2) that $J(t_h)$ is greater than $J(u)$ for all values of $u$ on the admissible side of $t_h$ and sufficiently near $t_h$. The same argument can now be made at each succeeding corner in turn.

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