**NULLIFYING FUNCTIONS**

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**Introduction.** A function \( f(x) \) defined on the unit interval \((0, 1)\) will be called nullifying if we can find a set \( S \) of \((0, 1)\) for which \( m(S) = 1, m(\{ f(x) : x \in S \}) = 0 \). Examples of homeomorphisms which are nullifying and hence termed singular are well known. We shall however consider simply the nullifying property itself.

If \( f(x) \) is nullifying and \( \phi(x) \) is not, one might expect that \( \phi(x) + f(x) \) shares with \( \phi \) the property of being not nullifying. But this is not always true as the following example shows. Let \( \alpha_1\alpha_2 \cdots \) denote the dyadic expansion for \( x \), that is, \( x = \alpha_1/2 + \alpha_2/2^2 + \cdots \) with \( \alpha_i = 0 \) or \( 1 \). Let \( v_1(x) = .\alpha_1\alpha_2\alpha_3 \cdots \) and \( v_2(x) = .0\alpha_2\alpha_4 \cdots \). It is easily verified that both \( v_1 \) and \( v_2 \) are nullifying. Hence \( f(x) = 1 - v_1 \) is also nullifying. Let \( \phi = x \). Then \( f + \phi = 1 - v_1 + x = 1 + v_2 \) is also nullifying.

But this suggests the question: Does there exist a nullifying function \( f(x) \) such that \( f(x) + \rho x \) is nullifying for every value of \( \rho \)? We construct such a function in the present note.

Our method of proof can be summarized as follows. Considering the set \( \{ f(x) + \rho x : x \in (0, 1) \} \), we let \( \rho = \cot \theta \). (Note \( \theta \neq 0 \).) If this set has measure zero, this will still be true if we multiply by \( \sin \theta \) and conversely. Thus we may consider the sets \( \{ x \cos \theta + f(x) \sin \theta : x \in (0, 1) \} \) for each \( \theta 
eq 0 \) between \(-\pi/2 \) and \( \pi/2 \). If we consider the line through the origin of inclination \( \theta \), we can assign a coordinate to each of its points in the usual manner with positive direction to the right or, in the case of the \( y \) axis, upwards. The set \( \{ x \cos \theta + f(x) \sin \theta : x \in (0, 1) \} \) is the set of coordinates of the projection onto this line of the graph of \( f(x) \). Thus it suffices to find a function \( f(x) \) which is such that the projection of its graph onto any line not parallel to the \( x \) axis is of measure zero. We proceed to find the graph of such a function by an intersection process on sets in the plane. This process is described in detail in what follows.

A more general question is: Given \( F(x, y, \rho) \), under what circumstances can we find a function \( f(x) \) such that \( F(x, f(x), \rho) \) is nullifying in \( x \) for every value of \( \rho \)? It is comparatively easy to abstract the properties of \( F = y + \rho x \) which are essential to the present discussion, and these will prove sufficient to obtain an answer to the question.

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However the writer hopes to obtain a more penetrating analysis of this subject soon. The writer will also consider the more general question of obtaining an $f(x)$ the substitution of which will make nullifying not only one but a set of $F$'s.

**Definition of a $G_x$.** Let us divide evenly the unit square of the plane in squares of side $1/2^x$. The coordinates of the vertices of these squares are dyadic rational, that is, of the form $a/2^x$ for some integer $a$. Two squares are said to be in the same column if they have the same projection on the $x$ axis. A set of such squares with one and only one (closed) square in each column will be termed$^8$ a $G_x$.

**Notation.** If $l_0$ is the line through the origin with inclination $\theta$ and $P$ is any point of the plane, we shall denote by $\pi_\theta(P)$ the projection of $P$ onto $l_0$. If $S$ is a set of points, we denote by $\pi_\theta(S)$ the projection of $S$ onto $l_0$. The linear measure of a linear set $S$ we denote by $\mu(S)$.

**Lemma 1.** Let $m$ be such that $0 \leq m \leq 1$. Let $\epsilon > 0$ and a $G_x$ be given. Let $\alpha$ be such that $\pi/2 \geq \alpha \geq \pi/4$ and $\cot \alpha = m$. Then we can find a $G_{x+t} \subset G_x$ such that for $0 \leq \theta \leq \pi/2$,

$$\mu(\pi_\theta(G_{x+t})) \leq (1 + m^2)^{1/2} \sin |\theta - \alpha| + \epsilon.$$

**Proof.** Let us consider a square $B$ of $G_x$. Then it is possible to take a line $l^{(1)}$ of slope $-m$, such that for every $x$ in the projection of $B$ on the $x$ axis, we have a point $(x, y)$ in $B$ and on $l^{(1)}$. Let $l^{(1)}$ intersect the left-hand side of $B$ at $P_B$ and the right-hand side at $Q_B$.

Let us divide $B$ into squares of side $1/2^{x+t}$. Now it is readily seen that for each column of smaller squares in $B$ there is at least one square with an interior point on $l^{(1)}$. (If $m = 0$, this is not true, but then we can substitute for “interior point,” “interior point or point on the upper side.”) Define $G_{x+t}$ so as to contain for each column the lowest such square.

We also take a square of side $1/2^{x+t}$, whose upper right-hand vertex is at $P_B$. We denote the lower left-hand end point of this square by $P_{B'}$. Similarly we take a square of side $1/2^{x+t}$ whose lower left-hand vertex is $Q_B$, and we denote the upper right-hand vertex by $Q_{B'}$.

We are going to consider $\pi_\theta(G_{x+t})$ and we let $<$ and $\leq$ refer to the order of the points on $l_0$, the direction of the greater being to the right of the smaller.

We shall show that if $P \in G_{x+t}$, and $0 \leq \theta \leq \alpha$, then $\pi_\theta(P_{B'}) \leq \pi_\theta(P)$.

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$^8$ We shall ignore the logical distinction between a set of squares and the corresponding set of points. $G_x$ denotes either a (closed) set of points or a set of squares according to the context.
For $P'_B (x'_B, y'_B)$ is the lower left-hand vertex of a square whose upper right-hand vertex alone is on $l^{(1)}$. Now every square $C$ of $G_{s+t}$ contains an interior point which is on $l^{(1)}$ (if $m = 0$, confer above). Thus if $l^{(1)'}$ is parallel to $l^{(1)}$ and through $P'_B$, every point $P (x, y)$ of $G_{s+t}$ is above or on $l^{(1)'}$. Furthermore $x > x'_B$.

Let $P' (x', y')$ be the intersection of $l^{(1)'}$ and the line through $P$ parallel to the $x$ axis. $P$ is above $P'$ and hence $\pi_\theta (P) \geq \pi_\theta (P')$. On the other hand $x' = x > x'_B$. Let $l$ denote the line through $P'_B$ perpendicular to $l$. The slope of $\bar{l}$ is more strongly negative than that of $l^{(1)'}$. It follows that for $x \geq x'_B$, $l^{(1)'}$ is to the right of $l$ and hence $\pi_\theta (P') \geq \pi_\theta (P'_B)$. This and the previous result yield $\pi_\theta (P) \geq \pi_\theta (P'_B)$. A similar argument will show that $\pi_\theta (Q'_B) \geq \pi_\theta (P)$.

Thus $\pi_\theta (P'_B) \geq \pi_\theta (P) \geq \pi_\theta (Q'_B)$. This implies that the projection of that part of $G_{s+1}$ which lies in $B$ is contained in the interval $\pi_\theta (P'_B) \pi_\theta (Q'_B)$. Now if $P_B$ has the coordinates $(x_B, y_B)$, then $Q_B$ has the coordinates $(x_B + 1/2^t, y_B - m/2^t)$, $P'_B$ has the coordinates $(x_B - 1/2^{t+1}, y_B - 1/2^{t+1})$, and $Q'_B$ has the coordinates $(x_B + 1/2^t + 1/2^{t+1}, y_B - m/2^t + 1/2^{t+1})$.

Consider $l_{\theta + \pi/2}$, the line through the origin perpendicular to $l$. According to a formula of elementary analytic geometry, the directed distance of any point $P (x, y)$ to $l_{\theta + \pi/2}$ is $x \cos \theta + y \sin \theta$, the directed distance being positive if $(x, y)$ is to the right of $l_{\theta + \pi/2}$ and negative to the left. This directed distance is not changed if we project $P$ onto $l$, and so $x \cos \theta + y \sin \theta$ represents also the directed distance of $\pi_\theta (P)$ from $l_{\theta + \pi/2}$, or since $l$ is perpendicular to $l_{\theta + \pi/2}$, from the origin $O$ along $l$.

Thus $O \pi_\theta (P'_B)$ has length $(x_B - 1/2^{t+1}) \cos \theta + (y_B - 1/2^{t+1}) \sin \theta$ and $O \pi_\theta (Q'_B)$ has the length $(x_B + 1/2^t + 1/2^{t+1}) \cos \theta + (y_B - m/2^t + 1/2^{t+1}) \sin \theta$. Hence $\pi_\theta (P'_B) \pi_\theta (Q'_B)$ has length

$$(1/2^t + 1/2^{t+1}) \cos \theta - (m/2^t - 1/2^{t+1}) \sin \theta$$

for $0 \leq \theta \leq \alpha$. Thus the projection of that part of $G_{s+t}$ which lies in $B$ has measure not greater than

$$1/2^t (\cos \theta - m \sin \theta) + 1/2^t (1/2^{t-1} (\cos \theta + \sin \theta))$$

$$< 1/2^t (\cos \theta - m \sin \theta) + 1/2^t (1/2^{t-2}).$$

There are $2^t$ squares like $B$, and this implies that for $0 < \theta \leq \alpha$, the projection of $G_{s+t}$ has measure not greater than

$$(\cos \theta - m \sin \theta) + 1/2^{t-2} = (1 + m^2)^{1/2} \sin (\alpha - \theta) + 1/2^{t-2}.$$

A similar argument holds in the case $\alpha < \theta \leq \pi/2$, with however the smaller added squares in different positions, and during the argument
certain inequalities are reversed. The corresponding result is that, for \( \alpha < \theta \leq \pi/2 \),

\[
\mu(\pi_\theta(G_{s+t})) < m \sin \theta - \cos \theta + 1/2^{t-2} = (1 + m^2)^{1/2} \sin (\theta - \alpha) + 1/2^{t-2}.
\]

If \( t \) is taken large enough so that \( 1/2^{t-2} < \epsilon \), the final results of the preceding two paragraphs are sufficient to prove our lemma.

**Lemma 2.** Suppose \( m > 1 \), and that a \( G_s \) and an \( \epsilon > 0 \) are given. Let \( \alpha \) be such that \( \pi/4 > \alpha > 0 \), \( \cot \alpha = m \). Then we can find a \( G_{s+t} \subset G_s \) such that for \( 0 \leq \theta \leq \pi/2 \),

\[
\mu(\pi_\theta(G_{s+t})) \leq (1 + m^2)^{1/2} \sin |\theta - \alpha| + \epsilon.
\]

**Proof.** Let \( B \) be any square of \( G_s \). Let \( l^{(1)}, l^{(2)}, \ldots, l^{(k)} \) denote a set of lines of slope \( -m \) with the following properties: (a) each \( l^{(i)} \) contains an interior point of \( B \); (b) if \( P_B^{(i)}(Q_B^{(i)}) \) is the line segment \( B \cdot l^{(i)} \) (\( B \) is closed), \( P_B^{(i)} \) is on the left side of \( B \), \( Q_B^{(i)} \) is on the right side of \( B \), and for \( i = 1, 2, \ldots, k-1 \), \( P_B^{(i+1)} \) is on the upper side of \( B \), \( Q_B^{(i)} \) is on the lower side of \( B \), and \( Q_B^{(i)} \) has the same \( x \) coordinate as \( P_B^{(i+1)} \). The existence of such a set of lines is easily shown.

Let the coordinates of \( P_B^{(i)} \) be \((x_B^{(i)}, y_B^{(i)})\), those of \( Q_B^{(i)} \) be \((u_B^{(i)}, v_B^{(i)})\). From the above we see that \( x_B^{(1)} = a/2^t \), \( y_B^{(1)} = b/2^t + \rho/2^t \) where \( 0 < \rho \leq 1 \), \( u_B^{(1)} = a/2^t + \rho/m \cdot 2^t \), \( u_B^{(2)} = a/2^t + (\rho + 1)/m \cdot 2^t \), and in general \( u_B^{(i)} = a/2^t + (\rho + i - 1)/2^t \cdot m \), for \( i = 1, 2, \ldots, k-1 \), while \( u_B^{(k)} = (a + 1)/2^t \). From this it follows that

\[
(\rho + k - 2)/m \cdot 2^t < 1/2^t \leq (\rho + k - 1)/m \cdot 2^t
\]
or \( (\rho + k - 2) < m < (\rho + k - 1) \) or \( k - 1 < m + (1 - \rho) \leq k \). Hence if \( \lfloor m \rfloor \) denotes the largest integer less than or equal to \( m \), then \( k = \lfloor m \rfloor, \lfloor m \rfloor + 1, \) or \( \lfloor m \rfloor + 2 \). Thus \( k \leq m + 2 \).

We next divide the square \( B \) of \( S_s \) into smaller squares of side \( 1/2^{s+t} \). The lines \( l^{(1)}, \ldots, l^{(k)} \) have been chosen so that for each \( x \) in the projection of \( B \) on the \( x \) axis, there is at least one \( l^{(i)} \) which contains a point \((x, y)\) in \( B \). This can be used to show that for each column of smaller squares in \( B \), there is at least one smaller square which contains a point on some \( l^{(i)} \) and interior to \( B \). Define \( G_{s+t} \) so as to contain the lowest such square in each column.

For each \( i \), let us consider those squares of \( G_{s+t} \) which contain points of \( l^{(i)} \). An argument similar to that used in Lemma 1 will show that the projection of this part of \( G_{s+t} \) on \( l_{s+t} \) has measure not greater than

\[
(\text{length of } l^{(i)} \cdot B) \sin |\theta - \alpha| + 1/2^{s+t-2}.
\]
The sum of the lengths of the $K(B) \cdot B$ is $(1/2^a)(1 + m^2)^{1/2}$ and there are at most $[m] + 2$ of them. Thus

$$\mu(\pi e_i B) \leq (1/2^a)((1 + m^2)^{1/2} \sin | | \theta - \alpha | + ([m] + 2)/2^{t-2}).$$

Since there are $2^s$ such squares $B$, we have

$$\mu(\pi e_i (G_{s+1})) \leq (1 + m^2)^{1/2} \sin | | \theta - \alpha | + ([m] + 2)/2^{t-2}. $$

We can take $t$ sufficiently large to obtain our result.

This result is interesting only in the range $0 < \theta < 2\alpha$, for outside this range other methods give more effective inequalities.

The argument of Lemmas 1 and 2 can be modified to apply to the case in which $m$ is negative, with the following result.

**Lemma 3.** Suppose an $m < 0$, a $G_2$ and an $\epsilon > 0$ are given. Let $\alpha$ be such that $\cot \alpha = m$ and $-\pi/2 \leq \alpha < 0$. Then we can find a $G_{s+1} \subset G_2$ such that for $-\pi/2 \leq \theta < 0$,

$$\mu(\pi e_i (G_{s+1})) \leq (1 + m^2)^{1/2} \sin | | \theta - \alpha | + \epsilon.$$

**Lemma 4.** Let $m_1$ and $m_2$ be given with $m_2 > m_1 \geq 0$. Let $\alpha_i$ be such that $\pi/2 \leq \alpha_i > 0$, $\cot \alpha_i = m_i$ for $i = 1, 2$. Consider the function

$$F(\theta) = \min ((1 + m_2^2)^{1/2} \sin (\theta - \alpha_2), (1 + m_1^2)^{1/2} \sin (\alpha_1 - \theta))$$

for $\alpha_2 \leq \theta \leq \alpha_1$. Then for this range of $\theta$,

$$F(\theta) \leq (m_2 - m_1)/2(m_1 + 1)^{1/2}.$$

**Proof.** For $\alpha_2 \leq \theta \leq \alpha_1$, $(1 + m_2^2)^{1/2} \sin (\theta - \alpha_2)$ is increasing while $(1 + m_1^2)^{1/2} \sin (\alpha_1 - \theta)$ is decreasing. Also we readily see that if $\eta$ is such that

$$(1 + m_2^2)^{1/2} \sin (\eta - \alpha_2) = (1 + m_1^2)^{1/2} \sin (\alpha_1 - \eta)$$

then $F(\theta)$ is equal to the first expression for $\alpha_2 \leq \theta \leq \eta$ and to the second expression for $\eta \leq \theta \leq \alpha_1$. Hence $F(\eta)$ is the maximum value of $F(\theta)$ for the given range of $\theta$.

Expanding $\sin (\eta - \alpha_2)$ and $\sin (\alpha_1 - \eta)$, using the definition of $\alpha_i$, collecting terms in $\sin \eta$ and $\cos \eta$ and dividing by $\cos \eta$ yields

$$\tan \eta = 2/(m_1 + m_2).$$

The value of $F(\eta)$ is then seen to be

$$(m_2 - m_1)/2(1 + (m_1 + m_2)^2/4)^{1/2} \leq (m_2 - m_1)/2(1 + m_2^2)^{1/2}.$$
denote the \( i \)th term in this sequence by \( m_i \). Notice that the terms of this sequence can be grouped so that the first group contains 1, the second group 3, the third group 9, and the \( k \)th group \( 2^{2k-3}+1 \). The maximum value in the \( k \)th group is \( 2^{k-2} \) and the difference between any two adjacent terms in this group is \( 1/2^{k-2} \).

Let \( \{ \epsilon_i \} \) denote the sequence such that if \( i \) is a subscript of the \( k \)th group of the preceding paragraph, then \( \epsilon_i = 1/2^k \). Let \( G^{(0)} = G_0 \), the unit square itself, and \( G^{(i)} \) be defined as the \( G_{i+1} \), which results when either Lemma 1, 2, or 3 (depending on \( m_i \)) is applied to \( G^{(i-1)} \), \( m_i \) and \( \epsilon_i \).

**Lemma 5.** Let \( G^{(i)} \) be as above. Let \( \theta \) be such that \( \pi/2 \geq \theta > 0 \). Then \( \lim_{i \to \infty} \mu(\pi_{\theta}(G^{(i)})) = 0 \).

**Proof.** Let \( \epsilon > 0 \) be given. Take \( k \) such that \( \epsilon/2 > 1/2^{k-1} \) and \( 2^{k-2} > \cot \theta \). Then we can find an \( m_i \) in the \( k \)th group such that \( m_i > 0 \) and \( m_i \geq \cot \theta > m_{i+1} \).

Now since \( G^{(i)} \supset G^{(i+1)} \), we have \( \pi_{\theta}(G^{(i)}) \supset \pi_{\theta}(G^{(i+1)}) \) and \( \mu(\pi_{\theta}(G^{(i)})) \geq \mu(\pi_{\theta}(G^{(i+1)})) \). Thus \( \mu(\pi_{\theta}(G^{(i+1)})) \) is subject to the inequality which Lemmas 1 and 2 impose upon \( \mu(\pi_{\theta}(G^{(i)})) \). Thus

\[
\mu(\pi_{\theta}(G^{(i)})) \leq \min \left[ \left( 1 + m_i^2 \right)^{1/2} \sin \left( \theta - \alpha_i \right), (1 + m_{i+1}^2)^{1/2} \sin \left( \theta - \alpha_{i+1} \right) \right] + \epsilon_i, \quad (\cot \alpha_i = m_i).
\]

Since \( \alpha_i \leq \theta \leq \alpha_{i+1} \), Lemma 4 yields

\[
\mu(\pi_{\theta}(G^{(i+1)})) \leq (m_i - m_{i+1})/2(1 + m_{i+1}^2)^{1/2} + \epsilon_i < 1/2^{k-1} + 1/2^k < \epsilon.
\]

Also if \( j \geq i + 1, \ G^{(j)} \subset G^{(i+1)}, \) and we obtain

\[
0 < \mu(\pi_{\theta}(G^{(j)})) \leq \mu(\pi_{\theta}(G^{(i+1)})) < \epsilon.
\]

Thus we have shown that given an \( \epsilon > 0 \), we can find an \( i \) such that for \( j > i \) this equation holds. The lemma is now proved.

**Lemma 6.** Let \( G^{(i)} \) be as in Lemma 5. Let \( G = \bigcap G^{(i)} \). Then \( G \) is a non-empty closed set such that if \( 0 < \theta \leq \pi/2 \), then \( \mu(\pi_{\theta}(G)) = 0 \).

**Proof.** Since the \( G^{(i)} \)'s form a decreasing sequence of closed sets, their intersection is a non-empty closed set. Since \( G \subset G^{(i)}, \) \( 0 \leq \mu(\pi_{\theta}(G)) \leq \mu(\pi_{\theta}(G^{(i)})) \). Thus Lemma 5 now implies \( \mu(\pi_{\theta}(G)) = 0 \).

Results similar to those of Lemmas 5 and 6 hold for \( 0 > \theta > -\pi/2 \). The method of obtaining them should be clear from the preceding discussion and we merely state the final result as follows.

**Lemma 7.** If \( G \) is as in Lemma 6, then for \( 0 > \theta > -\pi/2, \mu(\pi_{\theta}(G)) = 0 \).

**Lemma 8.** \( G \) is the graph of a function \( y = f(x) \).
Proof. For $0 \leq a \leq 1$, let $p_a$ denote the line $x = a$. Then the sets $p_a \cdot G^{(i)}$ form a decreasing sequence of closed intervals on $p_a$, with one and only one point $(a, b)$ in common. Thus $p_a \cdot G$ consists of one and only one point $(a, b)$ and $b$ is a function of $a$.

Now, as we have pointed out in the proof of Lemma 1, for any point $P(x, y)$, $x \cos \theta + y \sin \theta$ is the directed distance of $\pi_\theta(P)$ along $l_\theta$ from the origin. Using Lemmas 6, 7, and 8, we obtain that for $\pi/2 \geq \theta > 0$ or $0 > \theta > -\pi/2$,

$$0 = \mu(\pi_\theta(G)) = \mu(\{x \cos \theta + y \sin \theta; (x, y) \in G\})$$
$$= \mu(\{x \cos \theta + f(x) \sin \theta; 0 \leq x \leq 1\})$$
$$= |\sin \theta| \mu(\{x \cot \theta + f(x); 0 \leq x \leq 1\}).$$

Letting $\rho = \cot \theta$ we obtain that for every value of $\rho$,

$$\mu(\{f(x) + \rho x; 0 \leq x \leq 1\}) = 0.$$

Since $f(x)$ is the limit of step functions, it is measurable, and we have proved the following theorem.

Theorem. There exists a measurable function $f(x)$ defined for $0 \leq x \leq 1$, such that for every value of $\rho$, $f(x) + \rho x$ is nullifying.