CERTAIN SELF-RECIPROCAL FUNCTIONS

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In 1932 and 1933 I [5, 6] gave some rules connecting different classes of self-reciprocal functions. The object of this note is to derive some new self-reciprocal functions with the help of those rules.

I will say that a function is $R_\nu$ if it is self-reciprocal for $J_\nu$ transforms, where $\nu > -1$.

I will make use of the following results given in the papers referred to:

If $f(x)$ is $R_\mu$, the functions $g(x)$ given by the following integral formulas are all $R_\nu$:

\[(i)\] 
\[g(x) = x^{(\nu-\mu+1)/2} \int_0^\infty y^{(\nu-\mu+1)/2} J_{(\mu+\nu)/2}(xy)f(y)dy,\]

\[(ii)\] 
\[g(x) = x^{(\mu-\nu+1)/2} \int_0^\infty y^{(\mu-\nu+1)/2} J_{(\mu+\nu)/2}(xy)f(y)dy,\]

\[(iii)\] 
\[g(x) = \int_0^\infty \frac{y^{\mu+1/2}f(xy)}{(1 + y^2)^{1+\mu/2+v/2}} dy,\]

\[(iv)\] 
\[g(x) = \int_1^\infty \frac{y^{1/2-\mu}f(xy)}{(y^2 - 1)^{1-\mu/2+v/2}} dy,\]

\[(v)\] 
\[g(x) = \int_0^1 \frac{y^{1/2+v}f(xy)}{(1 - y^2)^{1+\mu/2-v/2}} dy.\]

If, in (ii) we take the familiar $R_\mu$ function

\[(1)\] 
\[x^{\mu+1/2}e^{-x^2/2}\]

for $f(x)$, we get

\[g(x) = x^{(\mu-\nu+1)/2} \int_0^\infty y^{(\nu-\mu+1)/2} J_{(\mu+\nu)/2}(xy)f(y)dy \cdot y^{\mu+1/2}e^{-y^2/2}dy = x^{(\mu-\nu+1)/2} \int_0^\infty y^{2\mu/2-\nu/2-1}e^{-y^2/2} J_{\mu/2+\nu/2}(xy)dy.\]

Evaluating this integral by Hankel’s formula [7], we get

\[g(x) = x^{(\mu-\nu+1)/2} \frac{\Gamma(\mu + 1)(x/2^{1/2})^{\mu/2+\nu/2}}{2^{-\nu-4\mu/4+\nu/4} \Gamma(1 + \mu/2 + \nu/2)} e^{-x^2/2} \cdot \,_1F_1(\nu/2 - \mu/2; 1 + \mu/2 + \nu/2; x^2/2), \quad \mu > -1.\]

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This shows that the function
\[ x^{v+1/2}e^{-x^2/2}\, _1F_1(v/2 - \mu/2; 1 + \mu/2 + v/2; x^2/2), \quad (\mu > -1), \]
which is the same as
\[ x^{v+2n+1/2}e^{-x^2/2}\, _1F_1(-n; n + v + 1; x^2/2), \quad \nu + 2n > -1, \]
is \( R_v \).

**Particular cases.**

(a) Since, when \( n \) is a positive integer,
\[ L_n^{(\alpha)}(x) = \frac{\Gamma(1 + \alpha + n)}{n!\Gamma(\alpha + 1)}\, _1F_1(-n; \alpha + 1; x), \]
where \( L_n^{(\alpha)}(x) \) denotes the generalized Laguerre polynomial of order \( n \), it follows that the function
\[ x^{v+2n+1/2}e^{-x^2/2}\, L_n^{(\alpha)}(x^2/2), \quad (\nu + 2n > -1), \]
is \( R_v \) for positive integral values of \( n \).

This result was given by Howell [4].

(\( \beta \)) Since, when \( n \) is a positive integer,
\[ T_n^{(\alpha)}(x) = \frac{(-1)^n}{n!\Gamma(1 + \mu)}\, _1F_1(-n; 1 + \mu; x), \]
where \( T_n^{(\alpha)}(x) \) is Sonine’s polynomial of order \( n \), it follows that the function
\[ x^{v+2n+1/2}e^{-x^2/2}\, T_n^{(\alpha)}(x^2/2), \quad \nu + 2n > -1, \]
is \( R_v \) for positive integral values of \( n \).

(\( \gamma \)) If we put \( n = -\nu - 1/2 \), our function becomes
\[ x^{-\nu-1/2}e^{-x^2/2}\, _1F_1(\nu+1/2; 1/2; x^2/2). \]
As a particular case, when \( \nu \) is an integer, this becomes a constant multiple of
\[ x^{-\nu-1/2}e^{-x^2/4}D_{-\nu-1}(x). \]

(\( \delta \)) When \( n = 1/2 - \nu \), the function is
\[ x^{\nu+1/2}e^{-x^2/2}\, _1F_1(\nu-1/2; 3/2; x^2/2), \quad (\nu < 2). \]
If \( \nu \) is an integer, this is a constant multiple of
\[ x^{\nu+1/2}e^{-x^2/4}D_{-\nu+1}(x). \]

The functions (3) and (4) were given by Mitra [2].

Taking the same function (1) for \( f(x) \), we have, from (iii),
\[ g(x) = \int_0^\infty \frac{y^{\mu+1/2}e^{-y^2/2}(xy)^{\mu+1/2}}{(1 + y^2)^{1+\mu/2+\nu/2}} \, dy = x^{\nu+1/2} \int_0^\infty \frac{y^{2\mu+1+\nu}e^{-y^2/2}}{(1 + y^2)^{1+\mu/2+\nu/2}} \, dy. \]
Evaluating this integral by a formula given by Whittaker [8] we find that \( g(x) \) is a constant multiple of
\[
x^{(\mu+\nu+1)/2}e^{\gamma x^2/4} W_{-1/2-\delta/4-\nu/4,\nu-\mu/4}(x^2/2),
\]
where \(\mu > -1\).

Thus, we see that the function

\[
x^{\nu-2m-1/2}e^{\gamma x^2/4} W_{\gamma m-\nu-1/2, m}(x^2/2),
\]
where \(\nu - 4m > -1\), is \(R_v\).

This function was given by Bailey [1].

If we put the same function (1) for \(f(x)\) in (i) and evaluate the corresponding integral by Hankel's formula [7], we arrive back at the \(R_v\) function

\[
x^{\nu+1/2}e^{-x^2/2}.
\]

If, in (iv), we take the same function for \(f(x)\) we arrive at the function (6) again.

If we take the same function for \(f(x)\) in (v), we again arrive at the function (2).

References


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1 Recently, Dhar [3] has given another proof of the fact that this function is \(R_v\). The other \(R_v\) function \(x^{\nu+1/2}e^{\gamma x^2/4} W_{-1/2-\nu-1/2, \nu}(x^2/2)\) that he gives, is the same as (5) with \(-m\) written for \(m\).