ON THE SUPPORTING-PLANE PROPERTY OF A
CONVEX BODY

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In an earlier paper, the authors have shown that in a linear space $\mathcal{S}$ with an inner product, a set $\mathcal{M}$ which is closed and linearly connected is supported at a set of boundary points which is everywhere dense on the boundary of $\mathcal{M}$, and an example is given to show that such a set $\mathcal{M}$ may have boundary points through which no supporting plane exists. The purpose of this paper is to show that if a set, in addition to being linearly connected and closed, also possesses inner points, then it is completely supported at its boundary points. In (I), reference was made to a paper by Ascoli in which such a result was obtained in a separable space. We do not assume our space $\mathcal{S}$ to be separable. The definitions and results of (I) will be used in this paper.

A set $\mathcal{K}$, which is a proper subset of the space $\mathcal{S}$, will be called a convex body if it is linearly connected, closed, and possesses inner points. In the sequel $\mathcal{K}$ will always denote a convex body.

With reference to the set $\mathcal{K}$, there is associated with each point $x$ of the space $\mathcal{S}$ a nonnegative number $r(x)$: if $x$ is an inner point of $\mathcal{K}$, $r(x)$ is defined as the least upper bound of the radii of spheres about $x$ which do not contain points exterior to $\mathcal{K}$; for other points of $\mathcal{S}$, $r(x)$ is defined to be zero. We will call $r(x)$ the radius at the point $x$.

If $x_1$ is a point of $\mathcal{K}$, all points $x$ of the sphere $||x-x_1||\leq r(x_1)$ are points of $\mathcal{K}$.

**THEOREM 1.** Let $r_1$ and $r_2$ be the radii at the points $x_1$ and $x_2$, respectively, of the convex body $\mathcal{K}$. Then the radius $r$ at the point $x = x_1 + k(x_2 - x_1)$, \hspace{1cm} 0 \leq k \leq 1,

satisfies

$$r \geq r_1 + k(r_2 - r_1).$$

**PROOF.** Let $y = x + \rho u$, where $\rho = r_1 + k(r_2 - r_1)$ and $||u|| = 1$. The points $y_1 = x_1 + r_1 u$ and $y_2 = x_2 + r_2 u$ are points of $\mathcal{K}$. But from the definitions of $x$, $\rho$, and $y$, it follows that $y = y_1 + k(y_2 - y_1)$. Hence $y$, being on the segment joining $y_1$ and $y_2$, is also a point of $\mathcal{K}$. Consequently

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all points on the boundary of the sphere with radius $\rho$ and center $x$ are in $\mathcal{R}$. Since $\mathcal{R}$ is linearly connected, also all points within this sphere are in $\mathcal{R}$. Therefore $r \geq \rho$ and the theorem is established.

The following corollaries, which appear self-evident in ordinary space, can be shown to be direct consequences of the preceding theorem.

**Corollary 1.** Each point of the segment joining two inner points of $\mathcal{R}$ is an inner point of $\mathcal{R}$.

**Corollary 2.** If $x_0$ is a boundary point and $x_1$ an inner point of $\mathcal{R}$, then the points $x = x_1 + k(x_0 - x_1)$ are inner points of $\mathcal{R}$ for $0 \leq k < 1$, and exterior points for $k > 1$.

With reference to a given boundary point $x_0$ of the set $\mathcal{R}$, there is associated with each point $x$, other than $x_0$, of the space $\mathcal{S}$ a non-negative number $s(x)$, defined by:

$$s(x) = \frac{r(x)}{||x - x_0||}.$$

If $x$ is an exterior point or a boundary point of $\mathcal{R}$, other than $x_0$, $s(x)$ is equal to zero; if $x$ is an inner point of $\mathcal{R}$, $s(x)$ is positive; $s(x_0)$ is not defined.

It is also obvious that $s(x) \leq 1$, since $r(x) \leq ||x - x_0||$.

**Theorem 2.** Let $x_0$ be a given boundary point of the convex body $\mathcal{R}$, and let $x_t$ be given by

$$x_t = x_0 + tu, \quad \text{where} \ t > 0, \ ||u|| = 1.$$  

Then, for fixed $u$,

(a) $s(x_t)$ is a non-decreasing function as $t \to 0$; and

(b) $\lim_{t \to 0} s(x_t)$ exists.

**Proof.** In case there are no points of $\mathcal{R}$ given by (1), the theorem is obviously true, for then

$$s(x_t) = 0 \quad \text{for} \ t > 0, \quad \lim_{t \to 0} s(x_t) = 0.$$

In case there are points of $\mathcal{R}$ given by (1), let $x_1$ and $x_2$ be two points of $\mathcal{R}$ on (1) for parameter values $t_1$ and $t_2$, where $t_1 < t_2$; then we have

$$x_1 = x_0 + t_1 u, \quad x_2 = x_0 + t_2 u;$$

$$s(x_1) = \frac{r(x_1)}{t_1}, \quad s(x_2) = \frac{r(x_2)}{t_2}.$$  

\[\text{Since} \ s \text{ is a function of } x_0 \text{ as well as } x, \text{ a more explicit notation would be } s(x_0, x); \text{ but the simpler notation will suffice, inasmuch as the function is to be used in the sequel only with reference to a fixed boundary point } x_0.\]
But $x_1 = x_0 + (t_1/l_2)(x_2 - x_0)$, and hence, by Theorem 1, we have

$$r(x_1) \geq \frac{t_1}{l_2} r(x_2),$$

since $r(x_0) = 0$. Therefore, by (2), $s(x_1) \geq s(x_2)$.

This result establishes part (a) of the theorem. Since $s(x_t)$ cannot exceed one, obviously part (b) of the theorem is true.

Let $\Sigma$ be the unit sphere about $x_0$; and let $p_u$ be the point on $\Sigma$ given by $p_u = x_0 + u$, $\|u\| = 1$. Let $x_t = x_0 + tu$, $(0 < t < 1)$, be the segment joining $x_0$ to $p_u$; and let

$$\sigma(u) = \lim_{t \to 0} s(x_t).$$

We thus have a function $\sigma(u)$ uniquely defined at each point $p_u$ on the sphere $\Sigma$. Obviously, by its definition, we have

$$0 \leq \sigma(u) \leq 1.$$

Also $\sigma(u) = 0$ only if the segment joining $x_0$ to $p_u$ does not contain any inner points of $\mathcal{K}$. If the segment joining $x_0$ to $p_u$ contains inner points of $\mathcal{K}$, we have $\sigma(u) > 0$.

**Lemma 1.** Let $p_u$ and $p_v$ be two points on $\Sigma$, such that

$$p_u = x_0 + u, \quad p_v = x_0 + v, \quad v = -u.$$

Then at least one of the numbers $\sigma(u)$ or $\sigma(v)$ is equal to zero.

**Proof.** Assume $\sigma(u) > 0$; then the segment joining $x_0$ to $p_u$ contains inner points. Consequently, by Corollary 2, the segment joining $x_0$ to $p_v$ does not contain any inner points. Therefore, $\sigma(v) = 0$.

**Theorem 3.** Let $x_0$ be a given boundary point of the convex body $\mathcal{K}$, and let $\Sigma$ be the unit sphere about $x_0$. Let $p_u$ and $p_v$ given by

$$p_u = x_0 + u, \quad \|u\| = 1, \quad p_v = x_0 + v, \quad \|v\| = 1$$

be two distinct points on $\Sigma$, for which $\sigma(u)$ and $\sigma(v)$ are both positive. Then there exists a point $p_w$ distinct from $p_u$ and $p_v$ for which

$$\sigma(w) > \frac{1}{2} [\sigma(u) + \sigma(v)].$$

The limit was shown to exist in Theorem 2; we are denoting the value of this limit by $\sigma(u)$. It may be of interest to note that

$$\sigma(u) = \lim_{x \to x_0, \text{along } x_t = x_0 + tu} s(x) = \lim_{x \to x_0} \frac{r(x) - r(x_0)}{\|x - x_0\|} = r'_u(x_0)$$

is the directional derivative of $r(x)$ at $x_0$ in the direction $u$. 
Proof. Let $x_t$ and $y_t$ be points of $\mathcal{K}$ given by

$$x_t = x_0 + tu, \quad y_t = x_0 + tv,$$

for $0 < t < 1$, and let $((u, v)) = \lambda$. Then, certainly $|\lambda| \leq 1$. But if $\lambda = 1$, $u = v$, and $p_u$ and $p_v$ are not distinct. If $\lambda = -1$, $u = -v$, in which case not both of the numbers $\sigma(u)$ and $\sigma(v)$ can be positive, because of Lemma 1. Consequently, we have

$$-1 < \lambda < 1.$$

Let $z_t = \frac{1}{2}(x_t + y_t)$; then $z_t = x_0 + \xi tw$, where $||w|| = 1$ and $\xi = \frac{1}{2}(1 + \lambda)^{1/2}$. Thus

$$0 < \xi < 1.$$

We thus have a point $p_w$ on the sphere $\Sigma$ defined by $p_w = x_0 + w$. Now, $r(z_t) \geq \frac{1}{2}[r(x_t) + r(y_t)]$, by Theorem 1. Hence

$$s(z_t) = \frac{r(z_t)}{\xi t} \geq \frac{1}{2\xi}\left[\frac{r(x_t)}{t} + \frac{r(y_t)}{t}\right] = \frac{1}{2\xi}\left[s(x_t) + s(y_t)\right],$$

and

$$\lim_{t \to 0} s(z_t) \geq \frac{1}{2\xi} \lim_{t \to 0} [s(x_t) + s(y_t)],$$

from which

$$\sigma(w) \geq \frac{1}{2\xi} [\sigma(u) + \sigma(v)] > \frac{1}{2} [\sigma(u) + \sigma(v)].$$

Thus the theorem is established.

Let $\bar{\sigma}$ denote the least upper bound of the function $\sigma(u)$ as $p_u$ varies over the sphere $\Sigma$. Then, also $0 \leq \bar{\sigma} \leq 1$; and $\bar{\sigma} = 0$ is possible only for sets which do not have any inner points. For a convex body $\mathcal{K}$, we have $0 < \bar{\sigma} \leq 1$.

In the material which follows, it is to be understood that $x_0$ is a fixed boundary point of the convex body $\mathcal{K}$, $s(x)$ is defined relative to $x_0$, $\Sigma$ is the unit sphere about $x_0$, $\sigma(u)$ is the function defined above on the boundary of $\Sigma$, and $\bar{\sigma}$ the least upper bound of $\sigma(u)$ on $\Sigma$.

Theorem 4. If there is a point $p_u$ on $\Sigma$ for which $\sigma(u) = \bar{\sigma}$, this point is unique.

Proof. Suppose, if possible, that there were a second point $p_v$ for which $\sigma(v) = \bar{\sigma}$. Then, by Theorem 3, since $\bar{\sigma} > 0$, there would be a point $p_w$ for which
\[\sigma(w) > \frac{1}{2} [\sigma(u) + \sigma(v)] = \tilde{\sigma}.\]

But since no \(\sigma(w)\) can exceed \(\tilde{\sigma}\), there cannot be a second point \(p_v\) of the type described.

**Theorem 5.** Let \(p_u\) be a point on \(\Sigma\) for which \(\sigma(u) = \tilde{\sigma}\). If \(v\) satisfies the conditions \(\|v\| = 1\) and \(((u, v)) < 0\), then the points \(z_t = x_0 + tv, t > 0\), are exterior points of \(\mathcal{R}\).

**Proof.** Let \(p_u = x_0 + u, \|u\| = 1\), and \(p_v = x_0 + v, \|v\| = 1\); and let \(((u, v)) = -\lambda\), where \(\lambda > 0\). Assume, if possible, that there is a point \(z = x_0 + dv, d > 0\), belonging to \(\mathcal{R}\). Let \(w\) be the projection (defined in (I)) of \(z\) on the line through \(x_0\) and \(p_u\). Then

\[w = p_u + c(x_0 - p_u),\]

where

\[c = \frac{((z - p_u, x_0 - p_u))}{\|p_u - x_0\|^2} = ((z - x_0 - u, -u)) = ((dv - u, -u)) = 1 + \lambda d.\]

Hence,

\[w = p_u - (1 + \lambda d)u = x_0 - \lambda du.\]

On the segment joining \(x_0\) to \(p_u\), let \(x_t = x_0 + tu\) be an inner point of \(\mathcal{R}\). Let \(y_t\) be the projection of \(x_0\) on the line through \(x_t\) and \(z\). Then

\[y_t = x_t + k(z - x_t)\]

where

\[k = \frac{((x_0 - x_t, z - x_t))}{\|z - x_t\|^2} = \frac{((tu, dv - tu))}{\|z - x_t\|^2} = \frac{\lambda td + t^2}{d^2 + 2\lambda td + t^2},\]

since \(z - x_t = z - x_0 + x_0 - x_t = dv - tu\) and

\[\|z - x_t\|^2 = d^2 - 2td((u, v)) + t^2 = d^2 + 2\lambda td + t^2.\]

From the above value of \(k\), it is easily seen that \(0 < k < 1\), which means that \(y_t\) is a point of \(\mathcal{R}\). The following are easily established:

\[\|y_t - x_0\|^2 = \frac{t^2d^2(1 - \lambda^2)}{d^2 + 2\lambda td + t^2} \neq 0,\]

since \(\lambda \neq \pm 1\), and

\[\|z - w\|^2 = d^2(1 - \lambda^2).\]
From these, and previous results, we obtain
\[ \frac{\| x_t - x_0 \|^2}{\| y_t - x_0 \|^2} = \frac{\ell^2(d^2 + 2\lambda d + t^2)}{t^2d^2(1 - \lambda^2)} = \frac{d^2 + 2\lambda d + t^2}{d^2(1 - \lambda^2)} = \frac{\| z - x_t \|^2}{\| z - w \|^2}. \]

Therefore, we have
\[ \frac{\| x_t - x_0 \|}{\| y_t - x_0 \|} = \frac{\| z - x_t \|}{\| z - w \|}. \]

Now \( s(y_t) = \frac{r(y_t)}{\| y_t - x_0 \|} \) and \( s(x_t) = \frac{r(x_t)}{\| x_t - x_0 \|} \), where \( r(y_t) \) and \( r(x_t) \) denote the radii at the points \( y_t \) and \( x_t \), respectively. Also \( r(y_t) \geq (1 - k)r(x_t) \), by Theorem 1 and the definition of \( y_t \). Hence
\[ \frac{s(y_t)}{s(x_t)} = \frac{r(y_t)}{\| y_t - x_0 \|} \frac{\| x_t - x_0 \|}{r(x_t)} \geq (1 - k) \frac{\| x_t - x_0 \|}{\| y_t - x_0 \|} \frac{\| z - x_t \|}{\| z - w \|}, \]
the last equality being a consequence of (3).

But \( k = \frac{\| y_t - x_t \|}{\| z - x_t \|} \) and \( 1 - k = \frac{\| z - y_t \|}{\| z - x_t \|} \). Therefore, from (4), we have
\[ s(y_t) \geq \frac{\| z - y_t \|}{\| z - w \|} s(x_t). \]

Now,
\[ \lim_{t \to 0} \frac{\| z - y_t \|}{\| z - w \|} = \frac{d}{d(1 - \lambda^2)^{1/2}} = \frac{1}{(1 - \lambda^2)^{1/2}} > 1, \]
since \( z \) and \( w \) are independent of \( t \), while \( y_t \to x_0 \) as \( t \to 0 \). Therefore, from (5),
\[ \lim_{t \to 0} s(y_t) \geq \frac{1}{(1 - \lambda^2)^{1/2}} \sigma(u) > \sigma(u) = \tilde{\sigma}. \]

But this is impossible; hence the assumption that \( z \) was a point of \( \mathcal{K} \) is untenable.

**Theorem 6.** Let \( p_u \) be a point on \( \Sigma \) for which \( \sigma(u) = \tilde{\sigma} \). Then the plane
\[ \pi(x) \equiv ((u, x - x_0)) = 0 \]
is a supporting plane of \( \mathcal{K} \) through the boundary point \( x_0 \).

**Proof.** If the plane (6) were not a supporting plane, there would be
a point \( z \) of \( \mathcal{K} \) for which \( \pi(z) < 0 \). Let \( v = (z - x_0) / \|z - x_0\| \); then

\[
((u, v)) = \frac{\pi(z)}{\|z - x_0\|} < 0, \quad z = x_0 + \|z - x_0\| v.
\]

But, \( v \) satisfies the conditions of Theorem 5; therefore \( z \) must be an exterior point of \( \mathcal{K} \). Consequently, there cannot be a point \( z \) of \( \mathcal{K} \) for which \( \pi(z) < 0 \); and (6) is indeed a supporting plane.

**Theorem 7.** Let \( x_0 \) be a given boundary point of the convex body \( \mathcal{K} \), and let \( \Sigma \) be the unit sphere about \( x_0 \). There is a unique point \( p_\alpha \) on \( \Sigma \) for which \( \sigma(\alpha) = \alpha \).

**Proof.** We have only to show the existence of one point \( p_\alpha \) for which \( \sigma(\alpha) = \alpha \). The uniqueness of this point will follow from Theorem 4.

From the definition of \( \tilde{\sigma} \) it follows that for any preassigned \( \epsilon > 0 \), there exists a point on \( \Sigma \) for which the value of \( \sigma \) is greater than \( \tilde{\sigma} - \epsilon \). Choose a monotone decreasing sequence of positive numbers \( \{\epsilon_n\} \) with limit zero. Corresponding to each \( \epsilon_n \) there exists a point \( p_{\alpha_n} \) on \( \Sigma \) for which \( \sigma(\alpha_n) > \tilde{\sigma} - \epsilon_n \). We wish to show that the sequence of points \( \{p_{\alpha_n}\} \) on \( \Sigma \) converges.

Let \( p_{\alpha_n} = x_0 + \alpha_n, \|\alpha_n\| = 1 \), and \( p_{\alpha_m} = x_0 + \alpha_m, \|\alpha_m\| = 1 \). Then

\[
(7) \quad \|p_{\alpha_n} - p_{\alpha_m}\|^2 = 2 - 2((\alpha_n, \alpha_m)).
\]

Let \( w = \frac{1}{2}(1/\xi)(\alpha_n + \alpha_m), \) where \( \xi \) is so chosen that \( \|w\| = 1 \). Then we have

\[
(8) \quad \xi^2 = \frac{1}{2} \left[ 1 + ((\alpha_n, \alpha_m)) \right].
\]

Let \( p_w = x_0 + w \); from the proof of Theorem 3, we know that

\[
\sigma(w) \geq \frac{1}{2\xi} \left[ \sigma(\alpha_n) + \sigma(\alpha_m) \right] > \frac{1}{2\xi} \left[ 2\tilde{\sigma} - \epsilon_n - \epsilon_m \right].
\]

But \( \tilde{\sigma} \geq \sigma(w) \); hence \( \tilde{\sigma} > (1/\xi) \left[ \tilde{\sigma} - (\epsilon_n + \epsilon_m)/2 \right] \), from which

\[
\xi^2 > \left[ 1 - \frac{1}{2\tilde{\sigma}} (\epsilon_n + \epsilon_m) \right]^2.
\]

Using the value of \( \xi^2 \) from (8) we easily find that

\[
((\alpha_n, \alpha_m)) > 2 \left[ 1 - \frac{1}{2\tilde{\sigma}} (\epsilon_n + \epsilon_m) \right]^2 - 1.
\]

Then using (7), we obtain
Since \( \lim_{n,m \to \infty} \| p_{u_n} - p_{u_m} \| = 0 \) and the space \( \mathcal{S} \) is complete, as was assumed in (I) and throughout this paper, the sequence \( \{ p_{u_n} \} \) converges to a point \( p_u \). This point \( p_u \) is on \( \Sigma \), and moreover \( \sigma(p_u) = \tilde{\sigma} \), since it is easily shown that \( \sigma(p_u) \) is greater than \( \tilde{\sigma} - \epsilon \) for any pre-assigned positive \( \epsilon \).

**Theorem 8.** A convex body \( \mathcal{K} \) is completely supported at its boundary points.

**Proof.** Let \( x_0 \) be a boundary point of \( \mathcal{K} \). There exists a point \( p_u \) on the unit sphere \( \Sigma \) about \( x_0 \) for which \( \sigma(u) = \tilde{\sigma} \), by Theorem 7. Hence the plane \((u, x - x_0) = 0\) is a supporting plane of \( \mathcal{K} \) through \( x_0 \), by Theorem 6. Since similar statements can be made for each boundary point, \( \mathcal{K} \) is completely supported at its boundary points.

From the material above, the following additional result may be established without much difficulty:

**Corollary 3.** There exists a unique supporting plane through the boundary point \( x_0 \) of the convex body \( \mathcal{K} \) only if \( \tilde{\sigma} = 1 \); for \( \tilde{\sigma} < 1 \), there is an infinite number of supporting planes through \( x_0 \).

A primary classification of boundary points of a convex body may thus be made in terms of \( \tilde{\sigma} \), which is a function defined over the boundary of the convex body.

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