for an arbitrary original polygon $P$. Further, no other relations $R_{n-p} = 0$ ($p \neq p_1$ or $p_2 \cdots$ or $p_k$) are satisfied by $P'$ if $P$ remains general ($P'$ has no higher than the $k$th degree of regularity). This is also seen from (16'), where $\phi(\omega) \neq 0, R_{n-p} \neq 0$ (since $P$ is general); therefore $R_{n-p} \neq 0$.

In fact, no relations of any kind besides (18) are satisfied by $P' = MP$ if $P$ remains general. This is because, by the general theory of systems of linear equations, it can be readily shown that if the conditions (17) are satisfied by the coefficients $\alpha$ in (2), then the conditions (18) are sufficient as well as necessary in order that (2) be solvable for the $z'$s in terms of the $z''$s. This is to say that for any polygon $P'$ obeying (18) a polygon $P$ can be found such that $P' = MP$; indeed, the class of such polygons $P$ depends linearly on $k$ complex parameters.

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AXIOMS FOR MOORE SPACES AND METRIC SPACES

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We shall consider a set of five axioms in terms of the undefined notions of point and region. It will be shown that these axioms are independent and that they constitute a set of conditions necessary and sufficient for a space to be a complete metric space. It will also be shown that certain subsets of this set of axioms constitute necessary and sufficient conditions for a space to be (1) a metric space, (2) a Moore space, (3) a complete Moore space. Axiom 2 and a more general form of Axiom 1 have been stated by the author in an earlier paper [1]. Following terminology of F. B. Jones [2], a space is said to be a Moore space provided conditions (1), (2), and (3) of Axiom 1 (that is, Axiom 1a) of R. L. Moore’s Foundations of Point Set Theory [3] are satisfied. A space is said to be a complete Moore space provided it satisfies all the conditions of that axiom. Wherever the notion of region is employed, whether as a defined or an undefined notion, it is understood that a necessary and sufficient condition that a point $P$ be a limit point of a point set $M$ is that every region containing $P$ contain a point of $M$ distinct from $P$. The letter $S$ is used to denote the set of all points.

1 Presented to the Society, April 20, 1935, under the title Sets of independent axioms for complete Moore space and complete metric space.
Axiom 0. Every region is a point set.

Axiom 1. There exists a countable family $F$ such that (1) every element of $F$ is a collection of regions covering $S$, (2) if $R$ is a region and $A$ and $B$ are points of $R$, there exists a collection $G$ of $F$ such that if $g$ is a region of $G$ that contains $A$, $g$ is a subset of $(R - B) + A$.

Axiom 2. If $P$ is a point and $H$ and $K$ are regions containing $P$, there exists a region $R$ containing $P$ which is a subset both of the region $H$ and of $K$.

Axiom 3. If $\alpha$ is a monotone descending sequence of closed point sets $A_1, A_2, \ldots$ such that for each $n$ there exists a monotone descending sequence $\rho_n$ of distinct regions $R_1, R_2, \ldots, R_n$ containing $A_n$, then there exists a point common to all the elements of $\alpha$.

Axiom 4. If $G$ is a collection of regions covering $S$, there exists a collection $H$ of regions covering $S$ such that if $h_1$ and $h_2$ are intersecting regions of $H$, then $(h_1 + h_2)$ is a subset of a region of $G$.

Theorem 1. In order that a space be a Moore space, it is necessary and sufficient that it satisfy Axioms 0, 1, and 2.

The necessity of these conditions is evident. We shall undertake to show their sufficiency without changing the notion of region. Let $H_1, H_2, \ldots$ be a type $\omega$ sequence of all the elements of family $F$ postulated by Axiom 1. Let $G_1$ denote the collection of all regions $R$ such that $R$ is a subset of a region of $H_1$. Let $G_2$ denote the collection of all regions $R$ such that $R$ is a subset of a region of $H_1$ and a region of $H_2$. For each positive integer $n$ let $G_n$ denote the collection of all regions $R$ such that $R$ is a subset of a region of $H_i$ for each $i \leq n$. For each $n$, $G_n$ covers $S$, by Axiom 2. Furthermore, for each $n$, $G_n$ contains all the regions of $G_{n+1}$. The sequence $G_1, G_2, \ldots$ satisfies all the conditions of Axiom 1 of R. L. Moore.

As a means to proving the next theorem, we shall prove the following lemma on the basis of Moore’s Axioms 0 and 1:

Lemma 1. If $M$ is a set of points and $G$ is a collection of domains covering $M$, there exists a collection $H$ of domains covering $M$ such that no domain of $H$ is a subset of another domain of $H$ and such that every domain of $H$ is a subset of some domain of $G$.

Suppose that $M$ is a set of points and $G$ a collection of domains covering $M$. For each positive integer $n$ let $T_n$ denote the set of all points $P$ of $M$ such that some domain of $G$ contains every region of $G_n$ that contains $P$. Then $M = \sum_{n=1}^{\infty} T_n$. For each positive integer $n$
let $\theta_n$ denote a well-ordered sequence of the points of $T_n$. Let $\theta$ denote the sequence obtained by taking first the elements of $\theta_1$, then the elements of $\theta_2$, and so on. Let $t_{i,\mu}$ denote the first element of $\theta$, where $i$ is the smallest integer such that $t_{i,\mu}$ is an element of $\theta_i$ and where $\mu$ is an ordinal number denoting the order of $t_{i,\mu}$ in $\theta_i$. (Some sets $T_n$ may be vacuous.) We shall now define a sequence $\Delta$ of domains $D_1, D_2, \cdots$. Let $D_1$ denote the sum of all the regions of $G_i$ that contain $t_{i,\mu}$. Let $t_{i,\nu}$ denote the first point of $\theta$ not contained in $D_1$. Let $D_2$ denote the sum of all the regions of $G_j$ that contain $t_{i,\nu}$. In general, suppose that $\Delta_\alpha$ denotes any abschnitt of $\Delta$; then let $t_{k,\xi}$ denote the first point of $\theta$ not contained in any domain of $\Delta_\alpha$ and let $D_\alpha$ denote the sum of all the regions of $G_k$ that contain $t_{k,\xi}$. Let $H$ denote the collection of all the domains of $\Delta$. Then $H$ has the required properties.

**Theorem 2.** In order that a space be a complete Moore space, it is necessary and sufficient that it satisfy Axioms 0, 1, 2, and 3.

We shall first show the sufficiency of these conditions. Let $H_1, H_2, \cdots$ denote a type $\omega$ sequence of the elements of family $F$ of Axiom 1. For each positive integer $n$ let $G_n$ denote the collection of all regions $R$ such that $R$ is a point or a proper subset of a region of $H_n$ and of a region of $G_{n-1}$. It follows, with the help of Axiom 2, that sequence $G_1, G_2, \cdots$ satisfies conditions (1), (2), and (3) of Moore's Axiom 1. It remains to be shown that it satisfies condition (4). Suppose that $M_1, M_2, \cdots$ is a type $\omega$ sequence of nondegenerate closed point sets such that for each $n$, $M_n$ contains $M_{n+1}$ and is a subset of some region of $G_n$. Let $R_n$ denote a region of $G_n$ that contains $M_n$. Then $R_n$ is a proper subset of a region $R_{n-1}$ of $G_{n-1}$. Similarly $R_{n-1}$ is a proper subset of a region $R_{n-2}$ of $G_{n-2}$. Thus the conditions of Axioms 3 are satisfied and hence there exists a point common to all the sets $M_1, M_2, \cdots$.

We shall now show the necessity of these conditions by redefining region. Let $G_1, G_2, \cdots$ be a sequence of collections of regions postulated by Moore's Axiom 1. For each $n$ let $H_n$ denote a collection of domains covering $S$ such that no domain of $H_n$ is a subset of another domain of $H_n$ and such that every domain of $H_n$ is a subset of a region of $G_n$. For each $n$, $H_n$ exists, by Lemma 1. Let $F$ denote the family of all collections $H_n$. Let $H = \sum_{n=1}^\omega H_n$. If the domains of $H$ are called regions and if nothing else is called a region, then Axioms 0, 1, 2, and 3 are satisfied. (1) Clearly Axioms 0 and 1 are satisfied. (2) We shall show that Axiom 2 is satisfied. Let $h$ and $k$ denote two domains of $H$ having a point $P$ in common. There exists an integer $n$ such that
every region of \( G_n \) that contains \( P \) is a subset of \( h \cdot k \). Let \( R \) denote a domain of \( H_n \). Then \( R \) is a subset of some region of \( G_n \) and hence of \( h \cdot k \). (3) We shall now show that Axiom 3 is satisfied. Let \( \alpha \) denote a type \( \omega \) sequence of closed point sets \( A_1, A_2, \ldots \), and for each \( n \) let \( \rho_n \) denote a type \( n \) sequence of domains of \( H, R_1, R_2, \ldots, R_n \) satisfying the conditions of Axiom 3. Since for each \( n \) no domain of \( H_n \) is a subset of another domain of \( H_n \), it follows that there exists an \( i \geq n \) such that some domain of \( \rho_n \) belongs to \( H_i \) and hence is a subset of a region of \( G_i \). It follows that for each \( n, A_n \) is a subset of a region of \( G_n \) and hence there exists a point common to all the elements of \( \alpha \).

**Theorem 3.** In order that a metric space be complete it is necessary and sufficient that it satisfy Axiom 3.

This follows immediately with the aid of Theorem 2 and a result of J. H. Roberts [4] to the effect that every metric space that satisfies Axiom 1 of R. L. Moore is complete. In a metric space every interpretation of region that preserves the notion of limit point satisfies Axioms 0, 1, 2, and 4.

**Theorem 4.** In order that a space be metric it is necessary and sufficient that it satisfy Axioms 0, 1, 2, and 4.

We shall first show the sufficiency of these conditions. We have shown that Moore's Axiom \( 1_0 \) follows from Axioms 0, 1, and 2. If Axiom 4 be added, it can be shown that the following stronger analogue (due to R. L. Moore) of Moore's Axiom \( 1_0 \) follows: “There exists a sequence \( G_1, G_2, \ldots \) such that (1) for each \( n, G_n \) is a collection of regions covering \( S \), (2) for each \( n, G_n \) contains \( G_{n+1} \), (3) if \( R \) is a region and \( A \) and \( B \) are points of \( R \), there exists an integer \( n \) such that if \( h \) and \( k \) are two regions of \( G_n \) having a point in common and such that \( h \) contains \( A \), then \( h + k \) is a subset of \( (R - B) + A \).” Moore has shown that this proposition is a necessary and sufficient condition for a space to be metric.

We shall show the necessity of these conditions. Suppose that \( S \) denotes a space (\( D \)). Let all spheroids be called regions. Let collection \( H_n \) of family \( F \) be the set of all spheroids of radius less than \( 1/n \). Clearly Axioms 0, 1, and 2 are satisfied. We shall show that Axiom 4 is satisfied. Let \( G \) denote a collection of spheroids covering \( S \). For each positive integer \( n \) let \( T_n \) denote the set of all points \( P \) such that there exists a spheroid of \( G \) containing the sum of every two-linked chain of spheroids of radius less than \( 1/n \) that contains \( P \). Then \( S = \sum_{n=1}^{\infty} T_n \). Let \( Q_n \) denote the collection of all spheroids of radius
less than $1/n$ containing a point of $T_n$. Let $Q = \sum_{n=1}^{\infty} Q_n$. Then $Q$ is the required collection, for the sum of every two-linked chain of regions of $Q$ is a subset of some region of $G$.

**Theorem 5.** In order that a space be a complete metric space, it is necessary and sufficient that it satisfy Axioms 0, 1, 2, 3, and 4.

This is an immediate consequence of Theorems 3 and 4.

**Independence Examples**

**For Axiom 1.** Let $S$ be the set of all real numbers between 0 and 1. Let $p$ and $q$ denote two real numbers such that $0 < p < q < 1$. Let the collection of all regions be the collection of all segments $ab$ such that either (1) $0 < a < p$ and $q < b < 1$, or (2) $0 < a < p$ and $0 < b < q$, or (3) $q < a < 1$ and $q < b < 1$.

**For Axiom 2.** Let $S$ be the set of all points on the $x$ axis between $(-1, 0)$ and $(+1, 0)$. Let every segment of $S$ not containing $O (0, 0)$ or having $O$ as an end point be taken as a region. Furthermore, let every point set consisting of $O$ together with a segment of $S$ having $O$ as an end point be taken as a region. If $n$ is an odd positive integer, let collection $H_n$ of family $F$ of Axiom 1 be the collection of all regions not containing $O$ and of length less than $1/n$, together with all left-hand regions containing $O$ and of length less than $1/n$. If $n$ is even, we have the same statement except that we substitute right-hand regions containing $O$ for left-hand regions.

**For Axiom 3.** Let $S$ be the set of all rational points on the $x$ axis. Let the sets of all rational points of all segments be called regions.

**For Axiom 4.** Let $S$ be the set of all points on or above the $x$ axis. Let regions be the interiors of all circles lying wholly above the $x$ axis, together with all point sets $Q$ such that $Q$ is the interior of a circle tangent to the $x$ axis plus the point of tangency. (Example due to R. L. Moore.)

**References**


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