ON TOPOLOGICAL COMPLETENESS¹

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Recently A. Weil² defined a uniform space as a set of points \( p \) such that for each \( \alpha \) in a set \( A \) there is defined a set \( U_\alpha(p) \subset S \), the class of sets \( U_\alpha(p) \) satisfying the conditions:

I. \( \prod_\alpha U_\alpha(p) = \{ p \} \).

II. To each \( \alpha, \beta \in A \) there is a \( \gamma = \gamma(\alpha, \beta) \in A \) such that \( U_\gamma(p) \subset U_\alpha(p)U_\beta(p) \).

III. To each \( \alpha \in A \) there is a \( \beta(\alpha) \in A \) such that if \( p', p'' \in U_\beta(\alpha)(q) \), then \( p'' \in U_\alpha(p') \).

For the uniform space \( S \), Weil introduced the concept of Cauchy family \( \{ M_\beta \} \) of sets. Such a family is defined by the conditions that the intersection of any finite number of sets of the family is non-empty and that to each \( \alpha \in A \) there is a \( p_\alpha \in S \) and a \( \beta(\alpha) \) such that \( M_\beta(p_\alpha) \subset U_\alpha(p_\alpha) \). Weil gives a theory of completeness in these terms.

The writer has considered³ a space \( S \) of points \( p \) and neighborhoods \( U_\alpha(p) \) where \( \alpha \) is an element of a set \( A \) such that:

I. \( \prod_\alpha U_\alpha(p) = \{ p \} \).

II. To each \( \alpha \) and \( \beta \in A \) and \( p \in S \) there is a \( \gamma = \gamma(\alpha, \beta; p) \) such that \( U_\gamma(p) \subset U_\alpha(p)U_\beta(p) \).

III. To each \( \alpha \in A \) and \( p \in S \) there are \( \lambda(\alpha), \delta(p, \alpha) \in A \) such that, if \( U_\delta(p, \alpha)(q)U_\lambda(\alpha)(p) \neq \emptyset \), then \( U_\delta(p, \alpha)(q) \subset U_\alpha(p) \).

The uniformity conditions here are lighter than those in IIa and IIIa. A Cauchy sequence \( p_n \in S \) was defined by the condition that for every \( \alpha \in A \), \( n_\alpha \) and \( p_\alpha \in S \) exist such that \( p_n \in U_\alpha(p_\alpha) \) for \( n \geq n_\alpha \). \( S \) is complete if every Cauchy sequence has a limit. It was shown that there is a complete space \( S^* \) which contains a homeomorphic image of \( S \) such that the image of a Cauchy sequence in \( S \) is a convergent sequence in \( S^* \).

It is the object of this paper to show that Weil's space is a special case of the space \( S_{\text{III}a,\text{III}} \) and that the notion of Cauchy family in this space leads to the same theory of completeness as that previously developed.

Theorem 1. If \( S \) satisfies \( \text{III}_a \) and \( \beta^2(\alpha) = \beta(\beta(\alpha)) \), then from \( U_{\beta^2(\alpha)}(q)U_{\beta^2(\alpha)}(p) \neq \emptyset \) follows \( U_{\beta^2(\alpha)}(q) \subset U_\alpha(p) \).

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PROOF. Let \( s \leq U_{\beta(\alpha)}(q) U_{\beta(\alpha)}(p) \). Then from \( s \), \( q' \leq U_{\beta(\alpha)}(q) \) we have \( q' \leq U_{\beta(\alpha)}(s) \). Therefore \( U_{\beta(\alpha)}(q) \subseteq U_{\beta(\alpha)}(s) \). Similarly \( U_{\beta(\alpha)}(p) \subseteq U_{\beta(\alpha)}(s) \). Now from \( p \leq U_{\beta(\alpha)}(s) \) and \( U_{\beta(\alpha)}(q) \subseteq U_{\beta(\alpha)}(s) \) we have \( U_{\beta(\alpha)}(q) \subseteq U_{\alpha}(p) \).

COROLLARY. If \( S \) satisfies III, then \( S \) satisfies III.

PROOF. For any \( p \in S \) and \( \alpha \in A \) we need only take \( \lambda(\alpha) = \delta(p, \alpha) = \beta(\beta(\alpha)) \). The result is stronger than III since \( \delta(p, \alpha) \) is independent of \( p \).

From now on a space \( S \) is one satisfying I, II, III. A family of sets \( \{ M_{\beta} \} \) is a Cauchy family if the intersection of any finite number of \( M_{\beta} \) is non-empty and if for any \( \alpha \in A \) there is a \( \beta(\alpha) \) such that \( M_{\beta(\alpha)} \subseteq U_{\alpha}(p_{\alpha}) \) for some \( p_{\alpha} \in S \). We will say that \( S \) is W-complete if, for every Cauchy family \( \{ M_{\beta} \} \), \( \prod_{\beta} \overline{M}_{\beta} \neq 0 \), where \( \overline{M}_{\beta} \) is the closure of \( M_{\beta} \). We shall always use the notations \( \lambda(\alpha), \delta(p, \alpha) \) in the sense of III.

THEOREM 2. \( S \) is W-complete if and only if every Cauchy family \( \{ U_{\alpha}(p_{\alpha}) \} \) consisting of one \( U_{\alpha}(p_{\alpha}) \) for each \( \alpha \in A \) has the property that \( \prod_{\alpha} U_{\alpha}(p_{\lambda(\alpha)}) \neq 0 \).

PROOF. Assume \( S \) is W-complete. If \( \{ U_{\alpha}(p_{\alpha}) \} \) is a Cauchy family, then \( \prod_{\alpha} U_{\lambda(\alpha)}(p_{\lambda(\alpha)}) \neq 0 \). Since \( \overline{U}_{\lambda(\alpha)}(p) \subseteq U_{\alpha}(p) \) follows from III, the condition \( \prod_{\alpha} U_{\alpha}(p_{\lambda(\alpha)}) \neq 0 \) is necessary. Assume now that the condition is satisfied and let \( \{ M_{\beta} \} \) be a Cauchy family in \( S \). To each \( \alpha \in A \) there are \( p_{\alpha} \in S \) and \( M_{\beta(\alpha)} \) such that \( M_{\beta(\alpha)} \subseteq U_{\alpha}(p_{\alpha}) \). It is clear that \( \{ U_{\alpha}(p_{\alpha}) \} \) is a Cauchy family. Hence there is \( \rho \in \prod_{\alpha} U_{\alpha}(p_{\lambda(\alpha)}) \).

For any \( \gamma \in A \), consider the \( \lambda(\gamma) \) and \( \alpha = \delta(p, \gamma) \) of III. Since \( \rho \in U_{\lambda(\gamma)}(p_{\lambda(\gamma)}) \), \( U_{\lambda(\gamma)}(p_{\lambda(\gamma)}) \subseteq U_{\gamma}(p) \) and

\[
M_{\beta(\lambda(\gamma))} \subseteq U_{\lambda(\gamma)}(p_{\lambda(\gamma)}) \subseteq U_{\alpha}(p_{\lambda(\gamma)}) \subseteq U_{\gamma}(p).
\]

Hence

\[
0 \neq M_{\beta} M_{\beta(\lambda(\gamma))} \subseteq M_{\beta} U_{\gamma}(p)
\]

for all \( \beta, \gamma \). Thus \( \rho \in \prod_{\beta} \overline{M}_{\beta} \) and \( S \) is W-complete.

The space \( S^* \) referred to above is defined as follows. A family \( \{ U_{\alpha}(p_{\alpha}) \} \), one for each \( \alpha \in A \), such that for any \( (\alpha_1, \ldots, \alpha_n) \in A \), \( U_{\alpha_1}(p_{\alpha_1}) U_{\alpha_2}(p_{\alpha_2}) \cdots U_{\alpha_n}(p_{\alpha_n}) \neq 0 \) is denoted by II. We write II' \( \sim \) II'' if for every \( \alpha \in A \) there is a set \( (\alpha_1, \ldots, \alpha_n) \in A \) such that for some \( p_{\alpha} \in S \)

\[
\prod_{i=1}^{n} U_{\alpha_i}(p_{\alpha_i}') + \prod_{i=1}^{n} U_{\alpha_i}(p_{\alpha_i}'') \subseteq U_{\alpha}(p_{\alpha}).
\]
The relation $\Pi' \sim \Pi''$ classifies all $II$ into mutually exclusive classes $C(II)$ which are the points $P$ of $S^*$. The neighborhoods $U_{a_1 \cdots a_n}(P)$ are made up of all $Q = C(II^q)$ $\epsilon S^*$ where $\Pi^q = \{ U_a(q_a) \}$ is such that for some $\beta_i = \beta_i(\alpha_1, \cdots, \alpha_n; P)$, $i = 1, \cdots, m = m(\alpha_1, \cdots, \alpha_n; P)$,

$$
\prod_{i=1}^m U_{\beta_i}(q_{\beta_i}) \subset \prod_{i=1}^n U_{a_i}(p_{a_i}).
$$

The space $S^*$ (for which $A^*$ is the set of all finite subsets of $A$) satisfies I, II and III*.

**THEOREM 3.** $S^*$ is W-complete.

**PROOF.** We first show that if $\{ U_{a_1} \cdots a_n(P^{a_1 \cdots a_n}) \}$, where each $(\alpha, \cdots, \alpha) \in A$ occurs just once, is a Cauchy family in $S^*$, then

$$
\prod_{(\alpha_1, \cdots, \alpha_n) \in A} U^*_{a_1 \cdots a_n}(p^{a_1 \cdots a_n}) \neq 0.
$$

Consider the $U^*_a(P^\alpha)$ for all $\alpha \in A$. Now $P^\alpha = C(II^\alpha)$, $\Pi^\alpha = \{ U_{\gamma}(p^\gamma) \}$, $U_{\gamma}(p^\gamma)$ being a neighborhood in $S$ for each $(\alpha, \gamma) \in A$. Since, for each $(\alpha_1, \cdots, \alpha_n) \in A$, $\prod_{i=1}^n U_{a_i}^*(p^{a_i}) \neq 0$, we have

$$
\prod_{i=1}^n U_{a_i}(p_{a_i}) \neq 0.
$$

Hence the family $\Pi = \{ U_{\alpha}(p^\gamma) \}$ defines a $P = C(II) \epsilon S^*$. For any $(\gamma_1, \cdots, \gamma_m) \in A$

$$
\prod_{i=1}^m U^*_{\gamma_i}(P) = U^*_{\gamma_1 \cdots \gamma_m}(P).
$$

Both $U^*_{\gamma_i}(P)$ and $U^*_{\gamma_i}(P^{\gamma_i})$ are the set of all $Q = C(II^q)$, $\Pi^q = \{ U_{\beta}(q_{\beta}) \}$ such that for some $\beta_{ji} = \beta_j(\gamma_i)$, $j = 1, \cdots, k$, $\prod_{j=1}^k U_{\beta_j}(q_{\beta_j}) \subset U_{\gamma_i}(p^\gamma_i)$. Hence

$$
U^*_{\gamma_i}(P) = U^*_{\gamma_i}(P^{\gamma_i}), \quad U^*_{\gamma_1 \cdots \gamma_m}(P) = \prod_{i=1}^m U^*_{\gamma_i}(P^{\gamma_i}).
$$

Thus from the fact that $\{ U_{a_1} \cdots a_n(P^{a_1 \cdots a_n}) \}$ is a Cauchy family we have for any sets $(\alpha_1, \cdots, \alpha_n), (\gamma_1, \cdots, \gamma_m) \in A$
It follows that
\[ P \in \prod_{(a_1, \ldots, a_n) \in A} \overline{U_{a_1}^* \cdots a_n(P^{a_1 \cdots a_n})}. \]

Now suppose that \( \{ M_\beta^* \} \) is a Cauchy family of sets in \( S^* \). For every \( (\gamma_1, \ldots, \gamma_n) \in A \) there is a \( \beta(\gamma_1, \ldots, \gamma_n) \) such that for some \( P^{\gamma_1 \cdots \gamma_n} \in S^*, M_{\beta(\gamma_1, \ldots, \gamma_n)}^* \subseteq U_{\gamma_1}^* \cdots \gamma_n(P^{\gamma_1 \cdots \gamma_n}). \) Thus
\[ 0 \neq \prod_{i=1}^m M_{\beta(i_1, \ldots, i_n)}^* \subseteq \prod_{i=1}^m U_{\gamma_1}^* \cdots \gamma_n(P^{\gamma_1 \cdots \gamma_i}) \]
for every finite set \( (\gamma_1, \ldots, \gamma_n) \subseteq A \) and \( \{ U_{\gamma_1}^* \cdots \gamma_n(P^{\gamma_1 \cdots \gamma_n}) \} \) is a Cauchy family. Let \( P \in S^* \) satisfy (1). For any \( (\alpha_1, \ldots, \alpha_n) \subseteq A \) and \( P \in S^* \) let \( (\gamma_1, \ldots, \gamma_n) \subseteq A \) be the set of \( \gamma_i = \gamma_i(\alpha_1, \ldots, \alpha_n; P) \) satisfying III*. From III* and (1)
\[ U_{\gamma_1}^* \cdots \gamma_n(P^{\gamma_1 \cdots \gamma_n}) \subseteq U_{\alpha_1}^* \cdots \alpha_n(P). \]
Since \( \{ M_\beta^* \} \) is a Cauchy family, for any \( \beta \) and \( (\alpha_1, \ldots, \alpha_n) \subseteq A \) we have
\[ 0 \neq M_{\beta}^* M_{\beta}^*(\gamma_1, \ldots, \gamma_n) \subseteq M_{\beta}^*(\gamma_1, \ldots, \gamma_n) \subseteq U_{\gamma_1}^* \cdots \gamma_n(P^{\gamma_1 \cdots \gamma_n}) \subseteq U_{\alpha_1}^* \cdots \alpha_n(P), \]
\[ M_{\beta}^* U_{\alpha_1}^* \cdots \alpha_n(P) \neq 0. \]
Hence \( P \in \prod_{\beta} M_{\beta}^* \) and \( S^* \) is \( W \)-complete.

Theorem 4. If \( \{ M_\beta \} \) is a Cauchy family in \( S \) and \( f(S) \subseteq S^* \) is the homeomorphism defined above, then there is a \( P \in S^* \) such that
1. \( \prod_{\beta} f(M_{\beta}) = (P), \)
2. for any \( (\alpha_1, \ldots, \alpha_n) \subseteq A \) there are \( \beta_1, \ldots, \beta_m \) such that
\[ \prod_{i=1}^m f(M_{\beta_i}) \subseteq U_{\alpha_1}^* \cdots \alpha_n(P). \]

Proof. \( f(M_{\beta}) \) is the class of \( P = C(\Pi p) \) for all \( p \in M_{\beta} \) where for any \( (\alpha_1, \ldots, \alpha_n) \subseteq A \) and \( \lambda(\alpha_i) \) there are \( \beta(\lambda(\alpha_i)) \) and \( p_{\lambda(\alpha_i)} \) such that
\[ M_{\beta(\lambda(\alpha_i))} \subseteq U_{\lambda(\alpha_i)}(p_{\lambda(\alpha_i)}) \subseteq U_{\alpha_i}(p_{\lambda(\alpha_i)}), \]
\[ 0 \neq \prod_{i=1}^n M_{\beta(\lambda(\alpha_i))} \subseteq \prod_{i=1}^n U_{\alpha_i}(p_{\lambda(\alpha_i)}). \]
If \( q \in \prod_{i=1}^n M_{\beta(\lambda(\alpha_i))} \), then for \( \delta_i = \delta(p_{\lambda(\alpha_i)}, \alpha_i) \) we have
\[ q \in U_{\delta_1}(q) U_{\lambda(\alpha_1)}(p_{\lambda(\alpha_1)}), \quad U_{\delta_i}(q) \subseteq U_{\alpha_i}(p_{\lambda(\alpha_i)}), \quad i = 1, 2, \ldots, n. \]
Since

\[ q \in \prod_{i=1}^{n} U_{i}(q) \subseteq \prod_{i=1}^{n} U_{i}(\phi(\alpha_i)), \]

the family \( \{ U_{a}(\phi(\alpha_i)) \} \) defines a \( P = C(\Pi) \in S^* \) such that

\[ f(q) \in U_{y_1}^{*} \cdots u_{y_n}(f(q)) \subset U_{a_1} \cdots a_n(P). \]

Hence for such \( (\alpha_1, \ldots, \alpha_n) \in A \) there are \( P_{a_1} \cdots a_n = P \) and \( \beta_i = \beta(\lambda(\alpha_i)), i = 1, \ldots, n, \) such that

\[ 0 \neq \prod_{i=1}^{n} f(M_{\beta(\alpha_i)}) \subset U_{a_1} \cdots a_n(P). \]

Thus the family of all finite products \( \{ f(M_{\beta_1}) \cdots f(M_{\beta_n}) \} \) is a Cauchy family in \( S^* \) since
\[ f(M_{\beta_1}) \cdots f(M_{\beta_n}) = f(M_{\beta_1} \cdots M_{\beta_n}) \neq 0. \]
We have

\[ U_{a_1}^{*} \cdots a_n(P) f(M_{\beta}) \supset \prod_{i=1}^{n} f(M_{\beta(\alpha_i)}) f(M_{\beta}) \neq 0 \]
for all \( \beta \) and all \( (\alpha_1, \ldots, \alpha_n) \in A. \) In other words

\[ P \in \prod_{\beta} f(M_{\beta}). \]

But the intersection of all \( f(M_{\beta}) \) is \( P, \) for if \( P' \) is any other point there are sets \( (\alpha_1, \ldots, \alpha_n), (\alpha_1', \ldots, \alpha_n') \in A \) such that

\[ U_{a_1}^{*} \cdots a_n(P) U_{a_1}^{*} \cdots a_n(P') = 0 \]
and

\[ P' \in S^* - U_{a_1}^{*} \cdots a_n(P) \subset S^* - \prod_{i=1}^{n} f(M_{\beta(\alpha_i)}) \subset S^* - \prod_{\beta} f(M_{\beta}). \]

We conclude with the remark that if \( S \) is \( W \)-complete, \( S \) is complete. Suppose \( \rho_n \) is a Cauchy sequence in \( S. \) Let \( M_n = (\rho_n, \rho_{n+1}, \ldots). \) Then the intersection of any finite set of \( M_n \) is non-empty and for any \( \alpha \in A \) there is an \( n(\alpha) \) such that \( M_{n(\alpha)} \in U_{a}(\rho_{a}) \) for some \( \rho_a \in S. \) Thus \( \{ M_n \} \) is a Cauchy family. \( S \) being \( W \)-complete, there is a \( \rho \in \prod_{\alpha} M_\alpha. \) Now for \( \rho, \) any \( \alpha \in A, \lambda(\alpha) \) and \( \delta = \delta(\rho, \alpha), \) we have

\[ M_{n(\delta)} \subset U_{\delta}(\rho), \quad 0 \neq U_{\lambda(\alpha)}(\rho) M_{n(\delta)} \subset U_{\lambda(\alpha)}(\rho) U_{\delta}(\rho), \]
\[ M_{n(\delta)} \subset U_{\delta}(\rho) \subset U_{a}(\rho), \quad \rho_n \in U_{a}(\rho), \quad n \geq n(\delta) = n(\delta(\rho, \alpha)), \lim \rho_n = \rho. \]

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