A THEOREM CONCERNING CLOSED AND COMPACT POINT SETS WHICH LIE IN CONNECTED DOMAINS

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The purpose of this paper is to show that the following theorem holds in any space which satisfies Axioms 0, 1, and 2 of R. L. Moore's *Foundations of Point Set Theory*.²

If \( g \) denotes a point set, \( \bar{g} \) will be used to denote the set \( g \) together with all its limit points. For each positive integer \( n \), \( G_n \) will denote the collection \( G_n \) of Axiom 1.

**Theorem.** If \( M \) is a closed and compact subset of a connected domain \( D \), then there exists a compact continuum containing \( M \) and lying in \( D \).

**Proof.** For each point \( P \) of \( D \), there exists a region \( g_P \) of \( G_1 \) containing \( P \) such that \( \bar{g}_P \) is a subset of \( D \). By Axiom 2, there exists a connected domain \( d_P \) containing \( P \) which is a subset of \( g_P \). Let \( U_1 \) denote the collection of all domains \( d_P \) for each point \( P \) of \( D \). The point set \( M \) is closed and compact, and hence, by Theorem 22 of Chapter I, it is covered by a finite subcollection \( W_1 \) of \( U_1 \). By Theorem 77 of Chapter I, for each pair of domains \( x \) and \( y \) of \( W_1 \) there exists a simple chain \( xy \) whose links are domains of \( U_1 \) and whose first and last links are \( x \) and \( y \) respectively. Let \( V_1 \) denote the collection of all domains \( v \) such that for some two domains \( x \) and \( y \) of \( W_1 \), \( v \) is a link of the chain \( xy \). The sum of all the domains of the finite collection \( V_1 \) is a connected domain \( D_1 \). Similarly, there exists a finite collection \( V_2 \) of connected domains such that if \( v \) is any domain of \( V_2 \), then \( \bar{v} \) is a subset of some region of \( G_2 \) and of some domain of \( V_1 \), and such that the sum of the domains of \( V_2 \) is a connected domain \( D_2 \).

This process can be continued. Thus there exists an infinite sequence \( V_1, V_2, V_3, \ldots \) such that, for each \( n \), (1) \( V_{n+1} \) is a finite collection of connected domains such that if \( v \) is any one of them then \( \bar{v} \) is a subset of some region of \( G_{n+1} \) and of some domain of \( V_n \) and of \( D \), and (2) the sum of all the domains of \( V_n \) is a connected domain \( D_n \), containing \( M \).

By Theorems 79 and 80 of Chapter I, the set of all points common to all the sets of the sequence \( D_1, D_2, D_3, \ldots \) is a compact continuum, and it contains \( M \) and lies in \( D \).

A modification of this argument proves this theorem for a space which satisfies Axioms 0 and 1 and is locally arcwise connected.

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1 Presented to the Society, February 24, 1940.