CONTINUA OF FINITE DEGREE AND CERTAIN PRODUCT SETS

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The continua of finite degree have been studied and identified with certain well known classes of continua in a paper by G. T. Whyburn. The author has shown that the continua of finite degree are identical with the continua homeomorphic to a continuum of finite length. The object of the present note is to obtain other internal characterizations of these continua.

The symbol \( M \) will represent a (compact) continuum. The continuum \( M \) is said to be of finite degree at the point \( p \in M \) provided that to each \( \epsilon > 0 \) there corresponds an uncountable family of neighborhoods \( (U) \) of \( p \) such that (a) the diameter of each neighborhood is less than \( \epsilon \), (b) each \( F(U) \) is finite, where \( F(U) \) is the boundary of \( U \), and (c) for any pair of neighborhoods \( U \) and \( V \) either \( U \subset V \) or \( V \subset U \). If every point is of finite degree, the continuum \( M \) is said to be of finite degree. The characterization which we find most useful below is that a continuum \( M \) is of finite degree if and only if every subcontinuum contains uncountably many local separating points of \( M \).

It will be shown that the classes of continua defined by each of the following properties are identical with the continua of finite degree.

**Property N°.** \( M \) is locally connected and to each pair of closed, disjoint subsets \( A \) and \( B \) in \( M \) there corresponds a finite collection of disjoint, perfect sets \( H^1, H^2, \ldots, H^k \) such that any continuum \( K \) in \( M \) intersecting both \( A \) and \( B \) contains some \( H^i \).

**Property Q.** If \( K \) and \( K_i, (i = 1, 2, \ldots) \), are nondegenerate continua in \( M \) with \( \lim K_i = K \), there exists an integer \( n \) such that \( \prod_{i=1}^n K_i \) is an uncountable set.

It will be noted that the Property N° is highly analogous to Property N which characterizes the locally connected continua such that no true cyclic element has a continuum of condensation.\(^5\)

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\(^1\) Presented to the Society, December 29, 1939.

\(^2\) National Research Fellow.


\(^4\) See abstract 45-9-321, this Bulletin.

\(^5\) This concept is due to R. L. Moore. See his *Fundamental Point Set Theorems*, Rice Institute Pamphlets, vol. 23, no. 1, 1936.

THEOREM A. A continuum is of finite degree if and only if it has Property N°.

PROOF. A continuum of finite degree has the property that any pair of closed, disjoint subsets \( A \) and \( B \) in \( M \) can be separated by a finite number of points \( q_1, q_2, \ldots, q_n \) which are local separating points of degree two.\( ^3 \) Suppose \( M \) is of finite degree. Set \( d = \min \rho(q^i, q^j), i \neq j, \rho(q^i, A + B) \). Since \( q^i \) is of degree two, there exists an uncountable family of neighborhoods \([U(q^i)]\) such that (a) \( \delta(U) < \frac{1}{4} d \), (b) \( F(U) \) consists of at most two points, and (c) for any pair of distinct elements \( U \) and \( V \) of the family either \( U \subset V \) or \( V \subset U \). Set

\[
Y^i = \sum_{i} F(U(q^i)),
\]

\( i \) fixed. Let \( H^i \) be a perfect subset of \( Y^i \). There is thus determined a finite collection of disjoint, perfect sets \( H^1, H^2, \ldots, H^n \). Let \( K \) be any continuum in \( M \) intersecting both \( A \) and \( B \). Since \( M \) is of finite degree (thus hereditarily locally connected), \( K \) may be taken as an arc. Since \( K \) contains some \( q^i \), it contains the corresponding \( H^i \).

Conversely, suppose \( M \) has Property N°. To prove \( M \) is of finite degree it suffices to show that every nondegenerate subcontinuum in \( M \) contains uncountably many local separating points of \( M \). Let \( K \) be a continuum in \( M \) containing the distinct points \( x \) and \( y \). Set \( A = x, B = y \). Then by Property N° there exists disjoint, perfect sets \( H^1, H^2, \ldots, H^n \) such that \( K \supset H^1 \) (say). If \( K \cdot L \) is uncountable, where \( L \) is the set of local separating points of \( M \), our end is attained. If \( K \cdot L \) is countable, there is a point \( z^1 \in H^1(M - L) \). Set \( d = 1/2 \min \rho(x, H^i), \rho(y, H^i), \rho(H^i, H^j), i \neq j \). Let \( V \) be a region in \( M \) containing \( z^i \) which is of diameter less than \( d \). Set \( d^1 = \rho(z^1, F(V)) \). Let \( W \) be the \( \frac{1}{2} d^1 \) neighborhood of \( z^i \), that is, the set of all points \( b \) such that \( \rho(z^1, b) < \frac{1}{2} d^1 \). Let \( X^1 \) and \( Y^1 \) be the components of \( K - KW \) containing \( x \) and \( y \), respectively. Since \( X^1 \) and \( Y^1 \) each contain a point of \( V \) and \( z^i \in M - L \), there exists an arc \( s^1 \) in \( V - z^i \) joining \( X^1 \) and \( Y^1 \). Hence \( K^1 = X^1 + s^1 + Y^1 \) is a continuum from \( x \) to \( y \) not containing \( H^1 \). If \( K^1 \) contains no other \( H^i \), we have a contradiction to Property N°. If \( K^1 \supset H^2 \), then \( H^2 \subset X^1 + Y^1 \) on account of the manner of selection of \( s^1 \). Since \( K \cdot L \) is countable, \( (X^1 + Y^1)L \) is countable, hence there exists a point \( z^2 \in H^2(M - L) \). Using the same \( d \) as before, a continuum \( K^2 \) is constructed (with \( K^2 \) replacing \( K \) ) such that \( K^2 \) contains neither \( H^1 \) nor \( H^2 \) and \( K^2 \) intersects both \( A \) and \( B \). Performing these steps (with \( z^2 \) replacing \( z^1 \) ) will clearly give a continuum \( K^2 = X^2 + s^2 + Y^2 \) not containing \( H^2 \), and since \( s^2 \partial H^1 = 0 \) and \( X^2 + Y^2 \subset K^1 \), we have \( K^2 \) contains neither \( H^1 \) nor \( H^2 \). After a finite number of such steps a con-
Continuum $K^j$ is obtained which intersects both $A$ and $B$ and contains no $H^j$. This completes the proof that having Property $N^0$ is equivalent to being of finite degree.

**Theorem B.** A continuum is of finite degree if and only if it has Property $Q$.

**Proof.** Let $M$ be of finite degree. Since this implies that $M$ is hereditarily locally connected, it may be assumed without loss of generality that the continua $K_i$ such that $\lim K_i = K$ are arcs. Let the end-points of $K_i$ be $a_i$ and $b_i$. Suppose $\lim a_i = x$ and $\lim b_i = y$. Since $K$ by assumption is a nondegenerate continuum, we may take $x \neq y$.

In a continuum of finite degree the points of degree two are dense on every subcontinuum, hence let $z$ be a point of degree two in $K - (x + y)$. Let $X$, $Y$ and $Z$ be neighborhoods of $x$, $y$ and $z$ respectively, such that $(X, Z + X, Y + Y, Z) = 0$. To $Z$ there corresponds an uncountable family of neighborhoods $(U)$ such that $\overline{U} \subset Z$ and $F(U)$ consists of at most two points. There is a definite $V = V(z)$ such that $V \subset U$ for uncountably many $U$. Since $z \in \lim K_i$, there is an integer $n$ such that arcs $K_i$, $i \geq n$, intersect $V$, hence if we take $m \geq n$ so large that $a_i \in X$, $b_i \in Y$ for $i \geq m$, then $K_i$ must contain two boundary points of each $U$ for which $V \subset U \subset Z$. Thus $\bigcap_n K_i$ is uncountable.

To show that Property $Q$ implies finite degree we show that infinite degree implies Property non-$Q$. It is clear that a continuum containing a convergence continuum has Property non-$Q$, hence we need consider only hereditarily locally connected continua. Let $T$ be an arc in the continuum $M$ which contains only a countable number of the local separating points of $M$. For each positive integer $n$ let $W_n$ be the set of all points of $M$ at a distance not greater than $1/n$ from $T$. Let $M_n$ be the component of $W_n$ which contains $T$. Let $L_n$ be the set of all points which separate the end-points $a$ and $b$ of $T$ in $M_n$. Then in $M_n$ there exist arcs $s_n$ and $t_n$ from $a$ to $b$ such that $s_n t_n = a + L_n + b$. Set $K_{2n - 1} = s_n$ and $K_{2n} = t_n$. Then $\lim K_n = T$. But $\bigcap_n K_i$ is at most countable, no matter what positive integer $n$ is. For $K_{2n - 1}$ and $K_{2n}$ have only a countable number of common points, namely, $a + L_n + b$. Hence $M$ does not have Property $Q$.

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