The max $|I_n^{(n)}(x)|$ is attained at $x = \pm 1$ since $\theta_{k+1} - \theta_k \leq 2\pi/(2n + \alpha + \beta - 1)$ provided $\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{4}$ and $x_k = \cos \theta_k$. Using the second asymptotic formula and the fact that $n\theta_k \to j_k$ as $n \to \infty$ where $j_k$ is the $k$th positive zero of $J_{\beta-1}(x)$, we find that

$$|I_n^{(n)}(1)| \to (\frac{1}{2}j_k)^{\beta-2} |\Gamma(\beta)J_\beta(j_k)|^{-1}$$

as $n \to \infty$, $k$ constant,

$I_n^{(n)}(-1) \to 0$ which proves the theorem:

**Theorem 7.** Max $|I_n^{(n)}(x)| \to (\frac{1}{2}j_1)^{\beta-2} |\Gamma(\beta)J_\beta(j_1)|^{-1}$ as $n \to \infty$ (where $\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{4}$, $j_1$ is first positive zero of $J_{\beta-1}(x)$).

A similar result holds for $I_n^{(n)}(x)$ if $\beta$ is replaced by $\alpha$.

For Legendre polynomials ($\alpha = \beta = 1$) this limit is approximately 1.602. For $\alpha = \beta = \frac{1}{2}$ and $\alpha = \beta = \frac{3}{4}$ the limit of Theorem 7 is also an upper bound for max $|I_n^{(n)}(x)|$ and max $|I_n^{(n)}(x)|$ . Whether this is true, in general, remains unanswered.

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**AN INVARIANCE THEOREM FOR SUBSETS OF $S^n$**

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The purpose of this paper is to establish the following.

**INVARIANCE THEOREM.** Let $A$ and $B$ be two homeomorphic subsets of the $n$-sphere $S^n$. If the number of components of $S^n - A$ is finite, then it is equal to the number of components of $S^n - B$.

In the case when $A$ and $B$ are closed this theorem is a very well known consequence of Alexander's duality theorem and its generalizations. In our case we also derive our result as a consequence of a duality theorem. However, the duality is established only for the dimension $n - 1$.

Given a metric space $X$ we shall say that $\Gamma^k$ is a $k$-cycle in $X$ if there is a compact subset $A$ of $X$ such that $\Gamma^k$ is a $k$-dimensional convergent (Vietoris) cycle in $A$ with coefficients modulo 2. We shall write $\Gamma^k \sim 0$ if $\Gamma^k \sim 0$ holds in some compact subset of $X$. The homology group of $X$ obtained this way will be denoted by $\mathcal{H}^k(X)$; the corresponding connectivity number, by $\rho^k(X)$. The number $\rho^k(X)$ can be either finite or $\infty$.

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1 Presented to the Society, December 28, 1939.
Duality theorem. Let \( A \subset S^n \) and let \( z_0, z_1, \ldots, z_m \) belong to \( m + 1 \) different quasi-components of \( S^n - A \). There are \( m \) linearly independent (modulo 2) \((n-1)\)-cycles

\[
(1) \quad \Gamma_1^{n-1}, \ldots, \Gamma_m^{n-1}
\]

of \( A \) such that

\[
(2) \quad v(\Gamma_i^{n-1}, \gamma_j^0) = \delta_{ij}, \quad i, j = 1, \ldots, m,
\]

where \( \gamma_j^0 \) is the 0-cycle \( z_0 + z_j \) (consisting of the two points \( z_0, z_j \) each of them with coefficient 1) and \( v(\Gamma, \gamma) \) is the linking number.

In case \( S^n - A \) has only \( m + 1 \) quasi-components, the cycles (1) form a basis for \( 3\mathbb{C}^{n-1}(A) \).

PROOF. In case \( A \) is closed the theorem turns out to be a particular case of the generalized Alexander duality theorem.\(^3\) We shall prove our theorem for arbitrary sets \( A \) using the theorem for closed sets.

Since \( z_0, z_1, \ldots, z_m \) belong to \( m + 1 \) different quasi-components of \( S^n - A \) there is a decomposition \( S^n - A = A_0 + A_1 + \cdots + A_m \) such that \( z_i \in A_i \) and \( A_i \overline{A}_i + \overline{A}_i A_i = 0 \) for \( i \neq j \), \( i, j = 0, 1, \ldots, m \). Let \( B_0, B_1, \ldots, B_m \) be open disjoint sets such that \( A_i \subset B_i \) for \( i = 0, 1, \ldots, m \) and let \( B = S^n - (B_0 + B_1 + \cdots + B_m) \). Clearly \( B \) is a closed subset of \( A \) and \( z_0, z_1, \ldots, z_m \) belong to \( m + 1 \) different quasi-components (equals components) of \( S^n - B \).

Applying the duality theorem to the closed set \( B \) we obtain the cycles (1) satisfying (2). In order to prove that they determine linearly independent elements modulo 2 of \( 3\mathbb{C}^{n-1}(A) \) consider a cycle \( \Gamma^{n-1} = a_1 \Gamma_1^{n-1} + \cdots + a_m \Gamma_m^{n-1} \) where \( a_i = 0, 1 \). It follows from (2) that \( v(\Gamma^{n-1}, \gamma_j^0) = a_j \). Therefore \( \Gamma^{n-1} \sim 0 \) in \( A \) implies \( a_1 = \cdots = a_m = 0 \).

Suppose now that \( S^n - A \) consists of exactly \( m + 1 \) quasi-components. It follows that the sets \( A_0, A_1, \ldots, A_m \) are connected.

Let \( \Gamma^{n-1} \) be an \((n-1)\)-cycle of \( A \) contained in some closed set \( D \subset A \). Let \( E_i \) be the component of \( S^n - (B + D) \) containing \( A_i \) \((i = 0, 1, \ldots, m)\) and let \( E = S^n - (E_0 + E_1 + \cdots + E_m) \). It follows that (1°) \( E \) is a closed subset of \( A \), (2°) \( S^n - E \) consists of exactly \( m + 1 \) quasi-components (equals components), (3°) the points

\(^3\) Two points \( x_1, x_2 \in X \) belong to the same quasi-component of \( X \) if there is no decomposition \( X = A_1 + A_2 \) such that \( x_1 \in A_1, x_2 \in A_2 \) and \( A_1 A_1 + A_1 A_2 = 0 \). If the number of quasi-components of \( X \) is finite then every quasi-component is a component.

$z_0, z_1, \ldots, z_m$ belong to different quasi-components of $S^n - E$. (4°) the cycles (1) and $\Gamma^{n-1}$ are contained in $E$. According to the duality theorem for closed sets the cycles (1) form a basis for $\mathcal{C}^{n-1}(E)$. This implies the existence of $a_1, a_2, \ldots, a_m$ ($a_i = 0, 1$) such that

$$\Gamma^{n-1} \sim a_1 \Gamma_1^{n-1} + \cdots + a_m \Gamma_m^{n-1} \text{ in } E.$$  

This proves the theorem since $E \subseteq A$.

Given a metric space $X$ let the number $b_0(X)$ be defined as follows:

- $b_0(X) = 0$ if $X = 0$,
- $b_0(X) = m$ if $X \neq 0$ and $X$ has exactly $m + 1$ components,
- $b_0(X) = \infty$ if $X$ has an infinity of components.

Clearly the value of $b_0(X)$ remains unchanged if we replace in its definition components by quasi-components. The duality theorem implies therefore the following:

(I) For every subset $A$ of $S^n$ we have

$$p^{n-1}(A) = b_0(S^n - A).$$

(II) For every two homeomorphic subsets $A$ and $B$ of $S^n$ we have

$$b_0(S^n - A) = b_0(S^n - B).$$

The invariance theorem stated in the introduction follows directly from (II).

If $X$ consists of an infinity of components, then instead of taking $b_0(X) = \infty$ we could define $b_0(X)$ to be the cardinal number corresponding to the class of all components of $X$. Similarly $p^k(X)$ could be redefined as a cardinal number. But with these new definitions (I) and (II) are no longer true.⁴ In fact, let $A$ be a subset of $S^1$ such that $S^1 - A$ is closed and enumerably infinite, and let $B$ be a subset of $S^1$ such that $S^1 - B$ is perfect and non-dense. It is clear that $A$ and $B$ are homeomorphic, that $b_0(S^1 - A) = \rho^0(A) = \rho^0(B) = \aleph_0$, and that $b_0(S^1 - B) = 2^{\aleph_0}$.

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⁴ That (II) is no longer true was first pointed out to me by Dr. L. Zippin.