and our theorem is proved. We can also prove the following:

3.2. Corollary. A necessary and sufficient condition that
\[(A + B)C = AC + BC\]
for positive \(A, B,\) and \(C\) is that either \(C = 1,\) or \(1 < C < \omega\) and \(\alpha_0 \leq \beta_0,\)
or \(\omega \leq C\) and \(\alpha_0 + \gamma_0 < \beta_0 + \gamma_0.\)

This corollary follows quite easily from the reasoning found in the preceding section.

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THE DECOMPOSITION THEOREM FOR
ABELIAN GROUPS

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Let \(G\) be an abelian group such that \(p^k g = 0\) for all \(g \in G,\) \(p\) prime, \(k\) fixed. We prove \(G\) has a basis, that is, a set of elements such that each \(g \in G\) is uniquely expressible as a linear combination of elements of the set.

Theorem. There exists an ascending chain of sets \(B_i, 0 \leq i \leq k,\) of elements of \(G\) with the properties:
(i) Every element in \(B_i\) is of order greater than \(p^{k-i}.\)
(ii) The elements in \(B_i\) are completely linearly independent.
(iii) If the order of the element \(g\) in \(G\) is greater than \(p^{k-i},\) then there exists a (unique) linear combination \(z\) of elements of \(B_i\) such that the order of \(g - z\) is at most \(p^{k-i}.\)

Since we may choose as \(B_0\) the vacuous set, we may assume that the sets \(B_0, \cdots, B_s\) have already been constructed in such a way as to meet the requirements (i) to (iii). In order to construct \(B_{s+1}\) we adjoin to \(B_s\) any greatest subset \(C\) of \(G\) with the following properties.
(a) All the elements in \(C\) are of order \(p^{k-s}.\)
(b) The join \(B_{s+1}\) of the sets \(B_s\) and \(C\) is an independent set.

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1 Presented to the Society, April 6, 1940.
2 Unique in that the number of nonzero terms in an expression for \(g\) is unique and only the arrangement but not the respective values of the nonzero terms may differ in two expressions for \(g.\)
The set $B_{s+1}$ satisfies (i) and (ii). In order to prove (iii) let $h$ be any element in $G$ whose order is at least $p^{k-\varepsilon}$. If firstly the order of $h$ is exactly $p^{k-\varepsilon}$, then it follows from the conditions (a) and (b) that there exists an integer $m$ and a linear combination $y = \sum a_j y_j$ of elements in $B_{s+1}$ so that $0 \neq mh = y$. Assume without loss of generality that $m$ is a power of $p < p^{k-\varepsilon}$. Then $(p^{k-\varepsilon}/m)mh = 0 = \sum a_j(p^{k-\varepsilon}/m)y_j$. By (ii), $p^{k-\varepsilon}$ divides $a_j(p^{k-\varepsilon}/m)$, so that $a_j/m$ is an integer (all $j$). Therefore the order of $h - \sum (a_j/m)y_j$ is $m < p^{k-\varepsilon}$. If secondly the order of $h$ is greater than $p^{k-\varepsilon}$, then there exists a linear combination $z$ of elements in $B_s$ such that the order of $h - z$ is at most $p^{k-\varepsilon}$, and thus (as above) $y = \sum c_jy_j$ can be found with $y_j$ in $B_{s+1}$ for which $h - z - y$ has order $< p^{k-\varepsilon}$. We have shown that $B_k$ is a basis of $G$.

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