A NOTE ON A THEOREM OF RADÓ CONCERNING THE \( (1, m) \)
CONFORMAL MAPS OF A MULTIPLY-CONNECTED
REGION INTO ITSELF\(^1\)

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Let \( G_z \) denote a region in the \( z \)-plane and let \( w = f(z) \) be a function
of \( z \) defined for \( z \subseteq G_z \) which has the following properties: (1) \( w = f(z) \)
is analytic and single-valued for \( z \subseteq G_z \), (2) \( z \subseteq G_z \) implies that \( f(z) \subseteq G_z \),
(3) to each point \( w_0 \subseteq G_z \) there correspond \( m \) and only \( m \) points \( z_{0}^{(k)} \)
\((k = 1, 2, \cdots, m)\) contained in \( G_z \) such that \( f(z_{0}^{(k)}) = w_0 \) \((k = 1, 2, \cdots, m)\)
where following the usual convention we count the \( z_{0}^{(k)} \) according to
their multiplicities. Then \( w = f(z) \) is said to define a \((1, m)\) conformal
map of \( G_z \) onto itself. Such maps have been studied by Fatou\(^2\) and Julia\(^3\)
for the case where \( G_z \) is simply-connected, and by Radó\(^4\) who treated
multiply-connected regions as well. Among the results which Radó
established is the following theorem:

Let \( G_z \) be a region of finite connectivity \( p \) \((>1)\); then there exists no
\((1, m)\) conformal map of \( G_z \) onto itself for \( m > 1 \).

Let us remark with Radó that the theorem is no longer valid if
\( G_z \) is of infinite connectivity, as simple examples from the theory of
the iteration of rational functions show.\(^5\) Radó’s proof of the theorem
just cited is based on the possibility of mapping one-to-one and con­
formally a region of finite connectivity \( p \), none of the components of
its boundary reducing to points, onto a region of connectivity \( p \), the
boundary of which consists of \( p \) disjoint circles. Other types of canoni­
cal regions yield the same result, notably one due to Koebe.\(^6\) It is the
object of the present note to establish Radó’s theorem directly with­
out appeal to the possibility of mapping one-to-one and conformally
the region \( G_z \) onto a canonical region. Our tools are the theory of
iteration and a simple modification of Nevanlinna’s principle of har­
monic measure.\(^7\)

Let \( G_z \), the region we are going to study, have as its boundary \( p \)
\((>1)\) disjoint continua \( \Gamma_k \) \((k = 1, 2, \cdots, p)\). It is evident that we

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\(^1\) Presented to the Society, April 27, 1940, under the title A note on a theorem of
Radó.


\(^5\) G. Julia, Journal de Mathématiques Pures et Appliquées, 1918.


\(^7\) R. Nevanlinna, Eindeutige analytische Funktionen, chap. 3.
may assume without loss of generality that the $\Gamma_k$ are all closed Jordan curves. Further let $w=f(z)$ define a $(1, m)$ conformal map of $G_z$ onto itself. In Radó’s paper cited above, it is shown that if $w=f(z)$ defines a $(1, m)$ conformal map of $G_z$ onto itself, then whenever $z$ tends to the boundary of $G_z$, so does $f(z)$, in such a manner that whenever $z$ tends to a given component $\Gamma_k$ of the boundary, then $f(z)$ tends to one and only one component of the boundary $\Gamma_k$ where the index $l_k$ depends on $k$. Now the relation $k\mapsto l_k$ ($k=1, 2, \cdots, p$) defines a permutation of the indices 1, 2, $\cdots$, $p$.

We shall understand by $f_n(z)$, the $n$th iterate of $f(z)$, that function defined by the recursive relations

\[(A) \quad f_0(z) = z, \quad f_1(z) = f(z), \quad \cdots, \quad f_n(z) = f[f^{n-1}(z)].\]

By the definition of $w=f(z)$ it is clear that the definition given by (A) for $f_n(z)$ is meaningful. Since the relation $k\mapsto l_k$ defines a permutation of the indices 1, 2, $\cdots$, $p$, a suitably chosen power of this permutation is the identity. Hence for a properly chosen whole number $n_0$, $f_{n_0}(z)$ has the property that when $z$ tends to a component $\Gamma_k$ of the boundary of $G_z$, then $f_{n_0}(z)$ tends to exactly the same component $\Gamma_k$. Let us denote $f_{n_0}(z)$ by $F(z)$.

We are now in a position to demonstrate Radó’s theorem directly. By the harmonic measure of $z$ with respect to $\Gamma_k$, denoted by $\omega(z, \Gamma_k, G_z)$ we understand that harmonic function defined for $z \in G_z$ which is single-valued and bounded for $z < \partial G_z$ and further takes on the boundary value 1 on $\Gamma_k$ and the boundary value 0 on all the components of the boundary of $G_z$ exclusive of $\Gamma_k$.

The relations

\[(B) \quad \omega(F(z), \Gamma_k, G_z) \equiv \omega(z, \Gamma_k, G_z), \quad k = 1, 2, \cdots, p,\]

are an immediate consequence of the principle of the maximum for harmonic functions. For, as $z$ tends to a given component of the boundary of $G_z$, $F(z)$ tends to the same component. Hence $\omega(F(z), \Gamma_k, G_z)$ has the same boundary values as $\omega(z, \Gamma_k, G_z)$, hence the identity. To establish Radó’s theorem we need consider only one of these identities, say

\[(C) \quad \omega(F(z), \Gamma_1, G_z) \equiv \omega(z, \Gamma_1, G_z).\]

Let $z_0$ be a point lying in $G_z$, and let $\omega(z_0, \Gamma_1, G_z) = \lambda_0$ ($0 < \lambda_0 < 1$). Then our identity (C) implies

\[\omega(F_n(z_0), \Gamma_1, G_z) = \lambda_0,\]

\[8\quad \text{R. Nevanlinna, ibid.}\]
where \( F_n(z) \) is the \( n \)th iterate of \( F(z) \) for \( n = 1, 2, \cdots \). Hence if \( z \) is on a given level curve defined by \( \omega(z, \Gamma_1, G_z) = \lambda \) \((0 < \lambda < 1)\), \( F_n(z) \) is on the same level curve for \( n = 1, 2, \cdots \). This permits us to conclude that no limit function of the sequence \( \{ F_n(z) \} \) is a constant. From this fact we infer that \( w = F(z) \) defines a \((1, 1)\) conformal map of \( G_z \) onto itself, and hence so does \( w = f(z) \); that is, \( m \) cannot exceed one.

Suppose contrary to our assertion that \( m > 1 \). Then \( w = F(z) \) would define a \((1, m^{n_0})\) conformal map of \( G_z \) onto itself and \( w = F_n(z) \) would define a \((1, m^{n_0})\) conformal map of \( G_z \) onto itself. Let \( \{ F_{k_n}(z) \} \) be a convergent subsequence of \( \{ F_n(z) \} \) and let \( F_0(z) \) denote the limit function of this subsequence. A point \( w_0 \) has, under the map defined by \( w = F_{k_n}(z) \), \( m^{n_0} \) antecedents all of which lie on the level curve defined by \( \omega(z, \Gamma_1, G_z) = \omega(w_0, \Gamma_1, G_z) \). But by Hurwitz's theorem, for \( k_n \) sufficiently large \( F_{k_n}(z) - w_0 \) and \( F_0(z) - w_0 \) have the same number of zeros, and this is manifestly impossible. Hence \( m \) cannot exceed one.

It is interesting to note that the technique employed in the present proof of Radó's theorem, that is, a combined use of the theory of iteration and of an extended form of the principle of harmonic measure (the modification consisting in the fact that we did not require the continuity of \( F(z) \) on \( \Gamma_1 \)) may be applied to other problems in the theory of the conformal mapping of multiply-connected regions. In particular, such a technique permits us to conclude that the number of \((1, 1)\) conformal maps of a given multiply-connected region of finite connectivity \( p \), where \( p > 2 \), bounded by \( p \) disjoint continua is finite. Koebe's proof of this theorem is based on the fact that the region in question can be mapped one-to-one and conformally onto a canonical region. We shall not give in the present note the details of a direct proof of this theorem. Let us remark however that it is based on the fact that for \( p > 2 \) each of the harmonic functions \( \omega(z, \Gamma_k, G_z) \) \((k = 1, 2, \cdots, p)\) has precisely \( p - 2 \) \((>0)\) critical points. A study of the level curves containing these critical points yields the desired result.

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