PROOF OF A THEOREM OF HALL
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In the Journal of the London Mathematical Society for July, 1937, Mr. Philip Hall gave a proof of the theorem, “If a group G of order g has a subgroup of order m for every divisor m of g such that \((m, g/m) = 1\), then G is a soluble group.” The proof is a very simple one in contrast to the rather difficult proof of the converse theorem which Hall had published in the same journal for April, 1927. It seems worthwhile to give a simpler proof of this converse.

**Theorem.** If \(g\) is the order of a soluble group \(G\) and \(m\) is a divisor of \(g\) such that \(m\) and \(g/m\) are relatively prime, then \(G\) has a subgroup of order \(m\) and furthermore all the subgroups of order \(m\) in \(G\) are conjugate under \(G\).

**Proof.** Since the theorem is true by default for prime power groups, let us suppose that it is true for all soluble groups of orders less than \(g\) and use the method of complete induction. Then the theorem is true for an invariant subgroup \(G'\) of prime index \(r\) in \(G\).

If \(r\) divides \(g/m\), then \(G'\) has a subgroup of order \(m\). Since \(G'\) is invariant and \((m, r) = 1\), every element of order dividing \(m\) in \(G\) must be in \(G'\). Hence all the subgroups of order \(m\) in \(G\) must be in \(G'\) where by hypothesis they form a complete set of conjugates under \(G'\) and consequently a complete set of conjugates under \(G\).

If \(r\) divides \(m\), then \(G'\) has a complete set of conjugates of order \(m/r\). Since \(G'\) is invariant, the subgroups of order \(m/r\) in \(G'\) are a complete set of conjugates under \(G\). Let \(M'\) be one of these subgroups of order \(m/r\) in \(G'\).

If \(M'\) is the only subgroup of order \(m/r\) in \(G'\), then, it must be invariant in \(G\). Then the quotient group is of order \(rg/m\) and since \(r\) is a Sylow divisor of the order of this quotient group, it has a subgroup of order \(r\). Then \(G\) has a subgroup of order \(r \cdot m/r\) or \(m\). On the other hand if \(M'\) is not the only subgroup of order \(m/r\) in \(G'\), let it be one of \(k\) subgroups of order \(m/r\) in \(G'\). Then \(k\) divides the order \(g/r\) of \(G'\) and the normalizer of \(M'\) in \(G\) is of order \(g/k\) divisible by \(m\). Since this normalizer is a soluble group of order \(g/k\) less than \(g\), it has a subgroup of order \(m\).

There remains only to show, for the case \(r\) divides \(m\), that all the subgroups of order \(m\) in \(G\) are conjugate under \(G\). Let \(M\) be a subgroup of order \(m\). If there is no other subgroup of order \(m\) in \(G\), then the theorem is true by default. However, if \(M_1\) is another subgroup
of order $m$ in $G$ it will be shown that $M$ and $M_1$ are conjugate under $G$.

Let the crosscut of $M$ and $G'$ be $\Gamma$ of order $\gamma$ and the crosscut of $M_1$ and $G'$ be $\Gamma_1$ of order $\gamma_1$. Then since $G$ is generated by $M$ and $G'$ as well as by $M_1$ and $G'$ and since $G'$ is invariant under both $M$ and $M_1$ we have

$$(g/r)(m)/\gamma = g, \quad (g/r)(m)/\gamma_1 = g,$$

whence

$$\gamma = \gamma_1 = m/r.$$

Then since $\Gamma$ and $\Gamma_1$ are of order $m/r$, they are conjugate under $G'$. If $S^{-1}\Gamma_1 S = \Gamma$, then $S^{-1}M_1 S$ and $M$ have a common invariant subgroup $T$. If $\Gamma$ is invariant in $G$, the quotient group $G/\Gamma$ is of order $gr/m$. Since $r$ is a Sylow divisor, the subgroups $M/\Gamma$ and $S^{-1}M_1 S/\Gamma$ of order $r$ are conjugate under $G/\Gamma$ and hence $M$ and $S^{-1}M_1 S$ are conjugate under $G$ as was to be proved. If, however, $\Gamma$ is not invariant under $G$, its normalizer is a proper subgroup of $G$ containing $M$ and $S^{-1}M_1 S$ which are therefore conjugate under the normalizer of $\Gamma$ as was to be proved.