A family $F$ of non-intersecting curves filling a metric space is called regular if, in a neighborhood of any point $p$, it is homeomorphic with a family of straight lines. We have given in another paper a necessary and sufficient condition, which we shall call (A') (to be described below), that a family $F$ be regular. We shall prove in this note that the following condition is sufficient:

(A) Given any point $p$, and a direction on the curve through $p$, there is an arc $pq$ in this direction with the following property. For every $\epsilon > 0$ there is a $\delta > 0$ such that for any $p'$, with $\rho(p', p) < \delta$, there is an arc $p'q'$ of $C(p')$ such that

$$p'q' \subset V_\epsilon(pq), \quad q' \subset V_\epsilon(q).$$

The condition (A') is the same, except that after (1), we add:

(2) If $r'$ and $s'$ are on $p'q'$ and $\rho(r', s') < \delta$, then $\delta(r's') < \epsilon$.

From the present theorem it is clear that the families of curves recently defined by Niemitzki are regular.

To prove the theorem, suppose (A) holds, but (A') does not. Then the following is true:

(B) There is a point $p$, and a direction of the curve $C(p)$, such that for any arc $pq$ on $C(p)$ in this direction, there is an $\epsilon > 0$, such that for any $\delta > 0$, there is a point $p'$, with $\rho(p', p) < \delta$, such that for any $q'$ on $C(p')$,

$$\text{either } p'q' \subset V_\epsilon(pq), \text{ or } q' \subset V_\epsilon(q),$$

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1. Presented to the Society, April 27, 1940.
2. Annals of Mathematics, (2), vol. 34 (1933), pp. 244–270. We refer to this paper as RF. By RF, Theorem 7A, $F$ is regular as there defined. The converse is proved as follows. By Theorem 17A, there is a cross-section $S$ through $p$. In a neighborhood of $p$, the curves are orientable (this is easily seen, for instance, with the help of Theorem 9B). Choose an open subset $S'$ of $S$, and let $U$ be all points $q' = g'(q, \alpha), q \in S', |\alpha| < \epsilon$ (see RF, §15); $U$ is a neighborhood of $p$, expressed as the product of $S'$ and the open line segment $-\epsilon < \alpha < \epsilon$.

By a curve, we shall mean here the topological image of an open line segment or of a circle. We shall use $\rho(p, q)$ for distance, $\delta(A)$ for the diameter of the set $A$, and $V_\epsilon(A)$ for the set of all points $p$, $\rho(p, A) < \epsilon$. Let $C(p)$ mean the curve of $F$ through $p$.

3. V. Niemczyi, Recueil Mathématique de Moscou, vol. 6 (48) (1939), pp. 283–292. We mention two further papers in the subject: H. Whitney, Duke Mathematical Journal, vol. 4 (1938), pp. 222–226, showing that if the curves fill a region in 3-space, a cross-section may be chosen so as to be a 2-cell; W. Kaplan, Duke Mathematical Journal, vol. 7 (1940), pp. 154–185, studying families filling the plane.

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(4) or there are points \(r', s'\) on \(p'q'\) such that \(\rho(r', s') < \delta\), and \(\delta(r's') \geq \epsilon\).

Choose a point \(p\) and a direction on \(C = C(p)\), by (B). Choose \(q\) on \(C\) in this direction, by (A). Choose \(\epsilon > 0\) by (B). For each positive integer \(i\), choose \(\delta_i\) by (A), with \(\epsilon\) replaced by \(\epsilon/i\). Choose \(p_i\) by (B), with \(\delta\) replaced by \(\delta_i\). Choose \(q_i\) by (A), with \(p'\) replaced by \(p_i\); then

\[
\rho(p, q_i) \subset V_{\epsilon/i}(pq), \quad q_i \subset V_{\epsilon/i}(q).
\]

By (B), as \(\epsilon_i \leq \epsilon\), we may choose \(p'_i\) and \(q'_i\) on \(pq_i\) so that

\[
\rho(p'_i, q'_i) < \delta_i, \quad \delta(p'_i q'_i) \geq \epsilon.
\]

By (6), we may choose \(r_i\) on \(p'_i q'_i\) so that \(\rho(p'_i, r_i) \geq \epsilon/2\). By (5) and (6), we may choose a subsequence so that for some points \(p'\) and \(r\) on \(pq\),

\[
\rho(p'_i, q'_i) \rightarrow \rho(p), \quad q'_i \rightarrow p', \quad r_i \rightarrow r;
\]
then \(r \neq p'\). Say, for definiteness, that \(r\) is in the direction of \(p'\) from \(p\).

The set of such points \(r\) which are limits of such sequences \(\{r_{n_i}\}\) forms a closed set, which, by (5), is in \(p'q\); we shall let \(r\) be the point furthest from \(p'\). (It might be \(q\).)

Assuming that (A) holds for the point \(r\) and the direction away from \(p'\), we shall arrive at a contradiction. Choose a point \(s\) on \(C\) in this direction from \(r\), by (A). (If \(C\) is a closed curve, it might happen that \(s\) is on the arc \(pr\).) Choose \(r'\) and \(s'\) on \(C\) just behind and just in front of \(r\), so that \(r'\) is on neither \(pp'\) nor \(rs\), and \(s'\) is not on \(p'r\). We shall show that for any \(\epsilon' > 0\) there is an integer \(j\) and a point \(s_j\) on \(p'_i q'_i\) within \(\epsilon'\) of \(s'\); as \(s\) is in \(pq\), by (5), this will contradict the definition of \(r\), and thus prove the theorem.

Set

\[
4\eta = \min \left[\rho(r', rs), 2\epsilon'\right].
\]

Choose \(r_-, r_+, s_-, s_+\) on \(C\) in the order \(r_- r' r_+ rs_- s_+ s\), so that \(r_-\) is not in \(rs\) and \(s_+\) is not in \(pr\) (if \(C\) is closed), and so that

\[
r_-r'_+ \subset V_\eta(r'), \quad s_-s'_+ \subset V_\eta(s').
\]

Set

\[
2\epsilon'' = \min \left[\rho(pr_-, r_+q), \rho(rs_-, s_+s), \eta\right].
\]

Using \(r, s, \) and \(\epsilon''\), choose \(\delta'' > 0\) by (A). By (7), we may choose \(j\) so that

\[
\epsilon/j < \epsilon'', \quad \rho(p'_i, p') < \epsilon'', \quad \rho(q'_i, p') < \epsilon'', \quad \rho(r_j, r) < \delta''.
\]
By the choice of $\delta''$, we may choose $s^*$ on $C(r_i)$ so that

$$r_js^* \subset V_{\epsilon''}(rs), \quad s^* \subset V_{\epsilon''}(s).$$

As $r_js^*$ is a connected set, (11), (12) and (10) show that there is a point $s_i$ on it such that $\rho(s_i, rs_+s) \geq \epsilon''$; hence, by (12), $\rho(s_i, s_+s) < \epsilon''$, and by (9),

$$\rho(s_i, s') < \epsilon'' + \eta < 2\eta \leq \epsilon'.$$

By (12) and (8),

$$\rho(r', r_js_i) > 2\eta.$$  

By (5) and (11),

$$p'_i q'_i \subset V_{\epsilon''}(pq).$$

By (11), (10), (5) and (9), there are points $p^*_i$ in $p'_i r_i$ and $q^*_i$ in $r_i q'_i$ such that

$$\rho(p^*_i, r') < 2\eta, \quad \rho(q^*_i, r') < 2\eta.$$

By this and (14), the arc $r_is_i$ is contained in the arc $p^*_i r_i q^*_i \subset p'_i q'_i$. Hence $s_i \subset p'_i q'_i$, which, with (13), gives the contradiction.

Harvard University