Although some teachers may fail to find some of the familiar material, the text is ample for a first course. Moreover the exercises give much of the factual material of the missing topics.

The result (7.12) in exercise 2, p. 32 seems to be incorrect. Problem 17 on page 121 would have been more accurately expressed by speaking of an *analytic* function of a complex variable. The exercises 4 and 5 on page 140 are incorrect. The University of Chicago Press may well feel proud of this book in every respect. It is my hope that the time taken in its preparation will be repaid by its wide use. Written by a master of oral and written exposition it should receive wide acclaim.

VERNON G. GROVE

Lattice Theory. By Garrett Birkhoff. (American Mathematical Society Colloquium Publications, vol. 25.) New York, American Mathematical Society, 1940. 6+155 pp. $2.50.

This is the first book on the far-reaching subject of lattices. The author has succeeded in giving a comprehensive, yet not too terse, account of the theory of lattices and its relation to other branches of mathematics. The general plan is to devote six chapters to the abstract theory, and three to applications. Chapter I deals with partially ordered systems, Chapter II with lattices and their general properties; the important modular axiom is assumed in Chapter III, and the axiom of complementation is added in Chapter IV. Distributive lattices and Boolean algebras end the abstract theory in Chapters V and VI. Thus the author takes the reader through the most important general classes of lattices, by imposing successively restrictive conditions. The last three chapters apply lattice theory to function theory, logic and probability theory.

A partially ordered system is introduced in Chapter I as a set L with a binary relation $\geqq$ on L which is reflexive and transitive and has the property that $x \geqq y$, $y \geqq x$ implies $x = y$. Most important in connection with general partially ordered systems are the principle of duality, viz., that L together with the converse $\leqq$ of the relation $\geqq$ forms a partially ordered system, and a study of the so-called chain conditions, one or both of which are satisfied in many examples. The ascending chain condition asserts that no infinite sequence (a_i) with $a_i < a_{i+1}$ exists; the descending chain condition is dual. Many examples of a partially ordered system are cited, a few of which are the set of all subsets of a set, classes of distinguished subsets of a set, real numbers, integers (relative to the relation of divisibility), partitions, and topologies on a space.

Most of the familiar examples of a partially ordered system are lattices, i.e., there exist for any two elements x, $y \in L$ a gr.l.b. $x \cup y$ and l.u.b. $x \cap y$. Indeed, many examples form *complete* lattices, i.e., lattices in which every subset has a gr.l.b. and l.u.b. The prototype of complete lattices is the lattice of all subsets of a set having a given extensionally attainable property (E. H. Moore), i.e., all *closed* sets relative to an abstract closure operation $\overline{A}$ having the properties $\overline{A} \geqq A$, $\overline{\overline{A}} = \overline{A}$, and $\overline{A} \geqq \overline{B}$ when $A \geqq B$. The MacNeille solution to the problem of embedding a partially ordered system in a complete lattice is given, and a number of other topics are discussed, including interesting questions regarding free lattices. Finally, it is shown that there exists in a lattice an intrinsic topology defined wholly in terms of the order relation.

The modular axiom states that $a \leqq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$. Among the examples cited which satisfy the modular axiom are the set of all invariant subgroups of a group (with operators), the set of all subspaces of a linear space, the set of right (left) or two-sided ideals in a ring, and the linear subspaces of an abstract projective geometry. The main result of Chapter III is that modularity implies the Jordan chain condition, which states that two principal chains connecting the same elements of the lattice are equal in length.

Complemented modular lattices are those lattices with zero O and unit I in which each element a has at least one "complement" x such that $a \cup x = I$, $a \cap x = O$. The fundamental result in Chapter IV characterizes abstract projective geometries (in the sense of Veblen and Young) as complemented modular lattices which are *simple* (in the sense of abstract algebra). This result is due to the author. The generalization to "continuous geometries" due to von Neumann is briefly described.

Chapters V and VI contain a study of distributive lattices, i.e., those in which $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$, and boolean algebras, i.e., complemented distributive lattices. The representation theory, which connects abstract boolean algebras with algebras of sets, and the applications of boolean algebras to topology are due chiefly to Stone.

The application in Chapter VII of lattice theory to function theory is more in the nature of an abstract theory of linear spaces which are partially ordered. Applications to logic in Chapter VIII are quite natural; logics considered are Brouwerian logic, classical logic and quantum logic. The final chapter continues Chapter VIII: A probability functional on a boolean algebra is a positive functional $p(x)$ such that $p(x \cup y) + p(x \cap y) = p(x) + p(y)$, and $p(O) = 0$, $p(I) = 1$. Probability functionals are found to lie in a vector lattice; the properties of this lattice are investigated. The chapter closes with a consideration of ergodic theorems.

Perhaps the most striking features of the book are the homogeneity of presentation and the thoroughness with which so large a subject is covered in so small a space. At the end the author includes a list of seventeen unsolved problems, which should prove provocative to the reader. The reviewer was especially pleased to find the first lattice-theoretic treatment ever given of Moore's extensional attainability, a simple but extremely useful tool. The section on polarity in Chapter II is also worthy of notice because of its wide applicability. The book is remarkably free from errors, the only ones noted being a few trivial errors in typography.

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