ON THE ANALOGUE FOR DIFFERENTIAL EQUATIONS
OF THE HILBERT-NETTO THEOREM

RICHARD COHN

If

\[ F_1, \ldots, F_r \]

is a finite system of differential polynomials in the unknowns

\( y_1, \ldots, y_n \), and if \( G \) is a differential polynomial which is annulled
by every solution of the system \((1)\), some power \( G^p \) of \( G \) is a linear
combination of the \( F_i \) and their derivatives of various orders, with
differential polynomials for coefficients. This analogue of the Hilbert-
Netto theorem was proved by J. F. Ritt\(^1\) for forms with meromorphic
coefficients, and by H. W. Raudenbush\(^2\) for the case of coefficients
belonging to an abstract differential field. In these proofs it is shown
that the denial of the existence of the exponent \( p \), above, of \( G \) leads
to a contradiction; no constructive method for obtaining admissible
values of \( p \) is given. The object of the present note is to present a
new proof of the analogue, for the case of meromorphic coefficients,
which is entirely constructive and produces a definite \( G^p \) as described
above.

Our proof will be based on the considerations in Chapters V and
VII of A.D.E. In Chapter VII, the problem of obtaining \( G \) is reduced
to the problem of determining unity as a linear combination of the \( F_i \)
in \((1)\) and their derivatives, in the case in which \((1)\) has no solutions.
In Chapter V it is shown how to decide, in a finite number of steps,
whether or not \((1)\) has solutions. Our problem thus assumes the fol­
lowing form: \textit{Given that \((1)\) has no solutions, it is required to express
unity as a linear combination of the } \( F_i \) \text{ and their derivatives}.

We assume that \((1)\) has no solutions and proceed to examine the
algorithm developed in §§65–67 of A.D.E. Adjoining to \((1)\) a finite
number of linear combinations of the \( F_i \) and their derivatives, we
obtain a system \( \Sigma \), devoid of solutions, with a basic set

\[ A_1, \ldots, A_q \]

which has the property that the remainder of every form in \( \Sigma \) with
respect to \((2)\) is zero. If \((2)\) consists of a single form \( A \) which is an

\(^{1}\) Ritt, J. F., \textit{Differential Equations from the Algebraic Standpoint}, chap. 7, referred
to below as A.D.E. American Mathematical Society Colloquium Publications, vol. 14,
1932.

\(^{2}\) Raudenbush, H. W., \textit{Ideal theory and algebraic differential equations}, Transactions
of this Society, vol. 36 (1934), pp. 361–368.
element of \( \mathcal{J} \), the coefficient field of our forms, we secure immediately a representation of the type desired for unity. Let us suppose that this is not the case. Then (2), considered as a set of simple forms, cannot be a basic set of a prime system; if it were, (1) would possess solutions (A.D.E., §65). Thus there must exist, for some \( j \leq q \), an identity

\[
J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}}(SA_i - H_1H_2) - L_1A_1 - \cdots - L_{j-1}A_{j-1} = 0,
\]

where \( J_i \) is the initial of \( A_i, i = 1, \ldots, j-1 \); and where \( H_1 \) and \( H_2 \) are reduced with respect to \( A_1, \ldots, A_j \). Let \( \Lambda_i^{(k)} \), \( k = 1, \ldots, j+1 \), represent the systems \( \Sigma + J_1, \ldots, \Sigma + J_{j-1}, \Sigma + H_1, \Sigma + H_2 \), respectively. We treat each \( \Lambda_i^{(k)} \) as (1) was treated. The adjunction of a finite number of forms to any \( \Lambda_i^{(k)} \) produces a system \( \Sigma_k \), with no solutions, and with basic sets lower than (2) which furnish zero remainders for the forms in \( \Sigma_k^{(k)} \).

Let us suppose that each \( \Sigma_i^{(k)} \) contains an element of \( \mathcal{J} \) different from zero. We see on examining these systems that there exist relations, procurable by constructive methods,

\[
\begin{align*}
(4) & \quad 1 = P + M_0H_1 + M_1H_1' + \cdots, \\
(5) & \quad 1 = Q + N_0H_2 + N_1H_2' + \cdots, \\
(6) & \quad 1 = R_i + S_{ij}J_i + S_{ij}J_i' + \cdots, \quad i = 1, \ldots, j-1,
\end{align*}
\]

accents indicating differentiation, where \( P, Q \), and the \( R_i \) are linear in the \( F_i \) and their derivatives.

We equate to unity the product of the right-hand members of (4), (5), and the equations (6). If both sides of the resulting equation are raised to a sufficiently high power, determinable in advance, we secure, as Raudenbush has shown, a relation of the type

\[
1 = L + T_0V + T_1V' + \cdots,
\]

accents indicating differentiation, where \( L \) is linear in the \( F_i \) and their derivatives, and \( V = J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}}H_1H_2 \). From (3) we see that \( V \) can be obtained as a linear expression in the \( F_i \) and their derivatives. We thus have such an expression as we are seeking for unity.

If, on the other hand, some system \( \Sigma_i^{(h)} \) does not contain a nonzero element in \( \mathcal{J} \), we apply to it the entire process applied to \( \Sigma \). We form in this way systems with basic sets lower than those of \( \Sigma_i^{(h)} \). The systems thus formed for the various \( \Sigma_i^{(k)} \) receiving our present treatment will be called, with no attempt to describe their complete history, systems \( \Sigma_2 \). In each \( \Sigma_2 \) a basic set yields only zero remainders.

---

Let us suppose that each $\Sigma_2$ contains a nonzero element in $\mathcal{F}$. What precedes shows that, for each $\Sigma_1^{(j)}$ as above, unity is linear in the forms of $\Sigma_1^{(j)}$ and their derivatives; this, again, gives the expression which we are seeking for unity.

If there are $\Sigma_2$ which contain no nonzero element in $\mathcal{F}$, we give them the treatment which is now familiar. By §67 of A.D.E., we know that our process can continue for only a finite number of steps, so that the possibility of determining for unity an expression of the type desired is established.

Columbia University