MOMENT PROBLEM FOR A BOUNDED REGION

L. B. HEDGE

1. Introduction. In this paper a solution of the moment problem given by Hausdorff for a bounded interval is extended to any bounded region in euclidean $n$-space, under certain conditions on polynomial expansions over the region. The resulting solution is valid for the $n$-dimensional sphere, and includes the Hausdorff case as well as the known conditions on the “class” of Fourier and Fourier-Stieltjes series.

2. Definitions and notation. Let $n$ be a positive integer, fixed but arbitrary. $R^n$ will denote the euclidean $n$-space, $(x)$ and $(y)$ will stand for $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$, points of $R^n$, and $E$ a bounded, closed subset of $R^n$. $v, \tau, i, j, k,$ and $s$, will be used for non-negative integers, and $(k), (s),$ and so on, will denote ordered $n$-tuples of non-negative integers $(k_1, k_2, \cdots, k_n), (s_1, s_2, \cdots, s_n),$ and so on, $(k) = (s)$ will mean $k_i = s_i, i = 1, 2, \cdots, n.$ $(0)$ will mean $(0, 0, \cdots, 0),$ \{u_{(k)}(x)\} and \{v_{(k)}(x)\} will be two sequences of polynomials such that

$$U_{(0)}(x) = V_{(0)}(x) = \text{const.,}$$

and by $\int_{E} U_{(k)}(x)V_{(s)}(x)dx = \begin{cases} 0, & (k) \neq (s), \\ 1, & (k) = (s), \end{cases}$

and by $\int_{E} f(x, y)d\Phi(E)$ will be meant the Lebesgue-Stieltjes integral over $E$ of $f$ considered as a function of a point $(y). B$ will be used for any Borel set with $B \subseteq E.$

If $f$ is integrable over $E$ we define

$$\mathcal{S}(f, x) = \sum_{(k)} A_{(k)}V_{(k)}(x), \quad A_{(k)} = \int_{E} f(x)U_{(k)}(x)dx,$$

$$S(x, y) = \sum_{(k)} U_{(k)}(x)V_{(k)}(y).$$

Let $L_v$ for every $v$ be a partition of $R^n$ into two subsets, one closed and bounded. We write $(k) \in L_v$ to indicate that $(k)$ belongs to the

---

1 Presented to the Society, June 20, 1940.
3 See, for example, A. Zygmund, Trigonometrical Series, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86.
bounded subset defined by $L_v$, and require that for every $(k)$ there exist a $v$ such that $(k) \subseteq L_v$, and that $(k) \subseteq L_v$ shall imply $(k) \subseteq L_v^*$ for all $v' \geq v$. Now let

$$S_v(x, y) = \sum_{(k) \in L_v} U_{(k)}(x)V_{(k)}(y),$$

$$\mathbb{E}_v(f, x) = \sum_{(k) \in L_v} A_{(k)}V_{(k)}(x) = \int_E S_v(x, y)f(y)dy.$$

If $T: \|a_{i\ell}\|$ is any regular Toeplitz transformation, we write

$$T\mathbb{E}_v(f, x) = \int_E TS_v(x, y)f(y)dy = \int_E K_v(x, y)f(y)dy.$$ 

If $P$ is a polynomial in $(x)$ we denote by $\mu_{(m)}$ the expression resulting from the substitution of $\mu_{(m)}$ for $x_1, x_2, \ldots, x_n$ in $P$.

### 3. Moment Problem

A solution of the moment problem for the set $E$ is given in the following theorem:

**Theorem.** Given $\{U_{(k)}(x)\}$, $\{V_{(k)}(x)\}$, $\{L_v\}$, and $T$ satisfying the conditions above, and such that $TS_v(x, y) = K_v(x, y) \geq 0$ for all $(x, y) \in E$, and all $v$, and such that for any $f$ integrable over $E T\mathbb{E}_v(f, x) = f(x)$ for every $(x) \in E$ for which $f$ is continuous, and uniformly on $E$ if $f$ is continuous on $E$, then in order that a sequence $\{\mu_{(m)}\}$ be expressible in the form

$$\mu_{m_1, m_2, \ldots, m_n} = \int_E x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} d\Phi(E),$$

where $\Phi$ is completely additive, defined over at least all Borel sets of $R^n$, and with

1. $\int_E |d\Phi(E)| \leq M,$
2. $\Phi(B) \geq 0,$
3. $\Phi(B) = \int_B \phi(x)dx$ and with
   (3a) $\phi \in L^p_B$, $p > 1$,
   (3b) $\phi \in L_B$,
   (3c) $|\phi| \leq M$,
   (3d) $\phi \in C_B$,

it is necessary and sufficient that

1. $\int_E [\mu_{(y)} \{K_v(x, y)\}] \, dx \leq M$ for all $v$,
2. $\mu_{(y)} \{K_v(x, y)\} \geq 0$ for all $(x) \in E$ and all $v$,
3a) $\int_E [\mu_{(y)} \{K_v(x, y)\}]^p \, dx \leq M$ for all $v$,

---

4 Zygmund, loc. cit., pp. 40–43.
(3b) \( \lim_{r \to \infty} \int_E \mu(y) \{ K_r(x, y) \} - \mu(y) \{ K_r(x, y) \} \, dx = 0 \),
(3c) \( \mu(y) \{ K_r(x, y) \} \leq M \) for all \( (x) \in E \) and all \( v \),
(3d) \( \lim_{r \to \infty} \mu(y) \{ K_r(x, y) \} - \mu(y) \{ K_r(x, y) \} = 0 \) uniformly in \( (x) \in E \).

The proof in each of the six cases closely parallels that of Hausdorff. The proof is given for case (1) to indicate the modifications:

Necessity. We have

\[
|\mu(y) \{ K_r(x, y) \}| = \left| \int_E K_r(x, y) \, d\Phi(E) \right| \\
\leq \int_E K_r(x, y) \, d\Phi(E),
\]

\[
\int_E |\mu(y) \{ K_r(x, y) \}| \, dx \leq \int_E \left\{ \int_E K_r(x, y) \, dx \right\} \, d\Phi(E) \\
\leq C \int_E |d\Phi(E)| \leq M
\]

for

\[
K_r(x, y) = \sum_{j=0}^{\infty} \sum_{(k) \in L_j} a_{xj} U_{(k)}(x)V_{(k)}(y),
\]

\[
\int_E K_r(x, y) \, dx = \sum_{j=0}^{\infty} \sum_{(k) \in L_j} a_{xj} \int_E V_{(k)}(y) \int_E U_{(k)}(x) \, dx
\]

\[
= \sum_{j=0}^{\infty} a_{xj} \leq \sum_{j=0}^{\infty} |a_{xj}| \leq C.
\]

Sufficiency. Let

\[
\Phi_r(B) = \int_B \mu(y) \{ K_r(x, y) \} \, dx,
\]

\[
\int_E |d\Phi_r(E)| = \int_E |\mu(y) \{ K_r(x, y) \}| \, dx \leq M
\]

and, by a well known theorem of Helly, there is a subsequence \( \{ \Phi_r \} \) and a function \( \Phi \) such that \( \int_E |d\Phi(E)| \leq M \) and \( \Phi_r(B) \to \Phi(B) \), and also \( \int_E V_{(k)}(y) \, d\Phi_r(E) \to \int_E V_{(k)}(y) \, d\Phi(E) \) whence \( \mu(y) \{ V_{(k)}(y) \} = \int_E V_{(k)}(y) \, d\Phi(E) \), and \( \Phi \) is a solution.

4. Examples and conclusion. If \( E \) is the unit sphere in \( \mathbb{R}^n \), \{ \( U_{(k)}(x) \) \} and \{ \( V_{(k)}(x) \) \} may be taken as the normalized polynomials of Appell-
Didon,\(^5\) \((k) \in L_\nu\) to mean \(\sum_{i=1}^{n} k_i \leq \nu\), and \(T\) any \((C, r)\) with \(r \geq n + 1.\)\(^6\)

In particular, for \(n = 1\) this reduces to the Hausdorff solution for the unit interval. If \(E\) is the circumference of the unit circle we may set

\[
U_0(x) = V_0(x) = (2\pi)^{-1/2}, \quad \text{and, for } k > 0,
\]

\[
U_{2k}(x) = V_{2k}(x) = (\pi)^{-1/2} \cos k\theta, \quad U_{2k-1}(x) = V_{2k-1}(x) = (\pi)^{-1/2} \sin k\theta
\]

with \((s) \in L_\nu\) meaning \(s \leq 2\nu\), \(T\) any \((C, r)\) with \(r \geq 1.\)\(^7\) Sequences \(\{ U_{(k)}(x) \} \) and \(\{ V_{(k)}(x) \} \) can be constructed by the Schmidt process for any bounded region in \(R^n.\) It would be interesting to know whether regular Toeplitz transformations of the type required for the present theorem exist in general.

Brown University

---


\(^7\) L. Fejer's theorem. See, for instance, Zygmund, loc. cit., p. 45.