ON BIORTHOGONAL MATRICES

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Introduction. Consider the basis consisting of a number system \( \mathcal{A} \) of type \( D \), two general ranges \( \mathcal{A}^1, \mathcal{A}^2 \), and two positive hermitian matrices \( e^1, e^2 \). We introduce two binary relations for pairs of non-modular matrices. The matrices \( \kappa^{12}, \phi^{21} \) are said to be contraceding as to \( e^{1}, e^{2} \) in case \( \kappa^{12}, \phi^{21} \) are by columns of \( M(e^1), M(e^2) \) respectively and such that \( J^{2*} \kappa^{12} \mu^2 = J^{2*} \phi^{21} \mu^2 \) for every \( \mu^2 \) in the set \( M(e^1) \cap e^2 \). It is evident that when \( \kappa^{12} \) is of type \( SO_f(\epsilon_1) \), then the contracedence property implies that \( J^{2*} \kappa^{12} \phi^{21} = \epsilon_1 \) but not conversely. The main results are stated in Theorems 2 and 3. We next consider \( e^0_1, e^0_2 \) both idempotent as to \( e^1, e^2 \) respectively such that \( J^{1*} \kappa^{12} \mu^1 = J^{1*} \phi^{21} \mu^1 \). Suppose that \( \kappa^{12} \) is by columns of \( M(e^0_1) \) and \( \phi^{21} \) is by rows-conjugate of \( M(e^0) \). The spaces \( M(e^1) \) and \( M(e^2) \) are in one-to-one correspondence (denoted by \( \leftrightarrow \)) via the transformations \( J^{1*} \kappa^{12} \) and \( J^{2*} \phi^{21} \). The correspondences are orthogonal in the sense that the moduli of the corresponding vectors are preserved.²

1. Preliminary results. Consider the basis \( \mathcal{A}, \mathcal{A}^1, \mathcal{A}^2, e^1, \) and \( \kappa^{12} \) which is by columns of \( M(e^1) \). E. H. Moore's generalized Fourier processes give \( e^2 \equiv J^{1*} \kappa^{12} e^1 \) and \( e^1 \equiv J^{2*} \kappa^{12} e^1 \). The spaces \( M(e^1) \) and \( M(e^2) \) are in one-to-one correspondence (denoted by \( \leftrightarrow \)) via the transformations \( J^{1*} \kappa^{12} \) and \( J^{2*} \kappa^{12} \), and the correspondences are orthogonal in the sense that the moduli of the corresponding vectors are preserved.²

(A)³ Suppose that \( M_1(e^1) \leftrightarrow M_1(e^2) \) and \( M_2(e^1) \leftrightarrow M_2(e^2) \) via the transformations \( J^{1*} \kappa^{12}, J^{2*} \kappa^{12} \). Then \( M_1(e^1) \) is a subset of \( M_2(e^1) \) if and only if \( M_1(e^2) \) is a subset of \( M_2(e^2) \); \( M_1(e^1) \) is linearly \( J^{1*} \)-closed if and only if \( M_1(e^2) \) is linearly \( J^{2*} \)-closed; and \( M_1(e^1) \) is everywhere dense in \( M_2(e^1) \) if and only if \( M_1(e^2) \) is everywhere dense in \( M_2(e^2) \).

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¹ Presented to the Society, June 20, 1940.
³ For the demonstrations of the following results, see the author's forthcoming paper On non-modular matrices.
Let $\varepsilon$ be a positive hermitian matrix. (B) The class of all vectors $\mu^1$ modular as to $\varepsilon^1$ such that $J^1\kappa^{*21}\mu^1$ is in $\mathcal{M}(\varepsilon^2)$ will be denoted by $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$. The intersection of $\mathcal{M}(\varepsilon^2)$ and $\mathcal{M}(\varepsilon^2)$ will be denoted by $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$. (C) The set $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$ is identical with the class of vectors $J^1\kappa^{*21}\mu^1$ for all $\mu^1$ in $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$. Moreover, (D) the sets $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ and $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$ are in one-to-one (orthogonal) correspondence via the transformations $J^1\kappa^{*21}$ and $J^2\kappa^{12}$. (E) The set $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ is the linear extension of the sum of $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ and the orthogonal complement within $\mathcal{M}(\varepsilon^1)$ of $\mathcal{M}(\varepsilon^2)$. (F) The class $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ is everywhere dense in $\mathcal{M}(\varepsilon^1)$ if and only if $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ is everywhere dense in $\mathcal{M}(\varepsilon^2)$.

2. Contraceding pairs of matrices. The basis of the paper consists of a number system $\mathfrak{A}$ of type $D$, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices $\varepsilon^1$, $\varepsilon^2$.

**Lemma 1.** Suppose that $\kappa^{12}$ is by columns of $\mathcal{M}(\varepsilon^1)$. Let $\mathcal{M}_0(\varepsilon^1)$ be a subset of $\mathcal{M}(\varepsilon^1)$; then $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ is contained in (contains) $\mathcal{M}_0(\varepsilon^1)$ only if $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all $\mu^1$ in $\mathcal{M}_0(\varepsilon^1)$. The converses are valid provided that $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$.

**Proof.** The lemma follows from (D) and (A) in §1.

If $\phi^{21}$ is by columns of $\mathcal{M}(\varepsilon^2)$, we introduce the notations due to E. H. Moore:

$$\varepsilon_\phi = J^2 \phi^{*12} \phi^{21}, \quad \varepsilon_\phi = J^1 \phi^{*12} \phi^{21}.$$

**Lemma 2.** Suppose that $\kappa^{12}$, $\phi^{21}$ are by columns of $\mathcal{M}(\varepsilon^1)$, $\mathcal{M}(\varepsilon^2)$ respectively. Then $\mathcal{M}(\varepsilon^1, \kappa^* \varepsilon^2)$ is contained in (contains) $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$ if and only if $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all $\mu^1$ in $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$.

**Proof.** This lemma is a special instance of Lemma 1.

**Definition 1.** The matrices $\kappa^{12}$, $\phi^{21}$ are said to be contraceding as to $\varepsilon^1 \varepsilon^2$ in case $\kappa^{12}$, $\phi^{21}$ are by columns of $\mathcal{M}(\varepsilon^1)$, $\mathcal{M}(\varepsilon^2)$ respectively and such that $J^2\kappa^{12} = J^3\phi^{*12} \mu^2$ for every $\mu^2$ in the set $\mathcal{M}(\varepsilon^2 \cap \varepsilon^2)$.

**Theorem 1.** Suppose that $\kappa^{12}$, $\phi^{21}$ are contraceding as to $\varepsilon^1 \varepsilon^2$. Then

1. $\mathcal{M}(\varepsilon^1 \varepsilon^2) = \{J^2 \phi^{*12} \mu^2 \} \cap \mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$;
2. $\mathcal{M}(\varepsilon^1 \cap \varepsilon^2) \subset \mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$;
3. $\mathcal{M}(\varepsilon^1 \varepsilon^2) \subset \mathcal{M}(\varepsilon^1 \cap \varepsilon^2)$ if and only if every $\mu^1$ for which $J^1\mu^1 = 0^1$ is in $\mathcal{M}(\varepsilon^2)$.

**Proof.** Part (1) is a consequence of the fact that

$$\mathcal{M}(\varepsilon^1 \varepsilon^2) = \{J^2 \kappa^{12} \mu^2 \} \cap \mathcal{M}(\varepsilon^2 \cap \varepsilon^2).$$
By hypothesis, we may replace $J^2\phi^{*12}$ by $J^2\phi^{*12}$. Since $\phi^{*12}$ is modular as to $e^1_2e^2$ and $M(e^1_2k^*e^2)$ is a subset of $M(e^1_2)$, every vector of the form $J^2\phi^{*12}\mu^2$ must belong to $M(e^1_2 \cap e^1_2)$. This proves (1). Part (2) is obvious from (1). Part (3) follows from (1) and (E).

**Theorem 2.** Suppose that $\phi^{21}$, $\kappa^{12}$ are contracting as to $e^2_1e^2$. Then

(1) $O^1M(e^1_2) \subset M(e^1_2)$ if and only if $\kappa^{12}$ is complete by columns of $M(e^1)$;

(2) the following four assertions are equivalent: (i) $M(e^1_2k^*e^2) \subset M(e^1_2)$; (ii) $\kappa^{12}$ is complete by columns of $M(e^1)$ and $M(e^2 \cap e^2) \subset M(e^2)$; (iii) $\kappa^{12}$ is complete by columns of $M(e^1)$ and $\kappa^{12}$, $\phi^{21}$ are contracting as to $e^1_2e^2$; (iv) $M(e^1_2k^*e^2) \subset [J^2\phi^{*12}\mu^2] \mu^2$ in $M(e^2 \cap e^2)$.

If one of the four conditions in (2) is valid, then $M(e^1_2k^*e^2) = M(e^1_2 \cap e^2)$ and $M(e^2 \cap e^2) = M(e^2 \cap e^2) = M(e^2 \cap e^2)$.

**Proof.** If $\kappa^{12}$ is complete by columns of $M(e^1)$, then $O^1M(e^1_2) = [0^1]$, which is, of course, contained in $M(e^1)$. Conversely, every $\mu^1$ satisfying $J^1\mu^1k^{12} = 0^2$ is in the orthogonal complement within $M(e^1)$ of $M(e^1)$, and hence is in $M(e^1)$. Thus $J^1\phi^{21}\mu^1 = 0^2$. Since $\phi^{21}$ is complete by rows-conjugate of $M(e^1_2)$, we have $\mu^1 = 0^1$, proving that $\kappa^{12}$ is complete by columns of $M(e^1)$.

To prove (i) implies (ii), we observe by (E) that $M(e^1_2k^*e^2)$ is the linear extension of the sum of $M(e^1_2k^*e^2)$ and $O^1M(e^1_2)$. If (i) is valid, then by part (1) just proved, $\kappa^{12}$ is complete by columns of $M(e^1)$. By Lemma 2, the condition that $M(e^1_2k^*e^2) \subset M(e^1_2)$ is equivalent to

(a) $M(e^2 \cap e^2) \subset [J^1k^{21}\mu^1 | \mu^1 \in M(e^1 \cap e^1_2)]$.

By hypothesis, we may replace $J^1k^{*21}$ by $J^1\phi^{21}$. Since, by (D),

(b) $[J^1\phi^{21}\mu^1 | \mu^1 \in M(e^1 \cap e^1_2)] = M(e^2 \cap e^2)$,

we have (i)$\rightarrow$(ii). To prove (ii)$\rightarrow$(iii), consider any vector $\mu^2$ in $M(e^2 \cap e^2)$. If (ii) is valid, then, by (D) (cf. (b) above), there exists a vector $\mu^1$ in $M(e^1 \cap e^1_2)$ such that

(c) $\mu^2 = J^1\phi^{21}\mu^1$,

whence

$J^2\phi^{*12} = J^2 \phi^{*12} J^1\phi \mu^1 = J^1\phi \mu^1 = \mu^1$.

Now by hypothesis and (c), we have $\mu^2 = J^1k^{*21}\mu^1$. As $e^1_2 = e^1_1$, it follows that

$J^2k \mu^1 = J^2k \mu^1 = J^1k \mu^1 = \mu^1$,.
proving the condition (iii). By Theorem 1, we have (iii)→(iv). Since \( \phi^{*12} \) is modular as to \( e_1^*e_2 \), we secure (iv)→(i).

The final statement follows by Theorem 1.

**Theorem 3.** The matrices \( \phi^{21}, \kappa^{12} \) are contraceding as to \( e^2e^1_0e^1 \), \( \phi^{21} \) is complete by columns of \( M(e^1) \), and \( M(e^1\kappa^*e^2) \subset M(e^1) \) if and only if \( \kappa^{12} \), \( \phi^{21} \) are contraceding as to \( e^1_0e^2 \), \( \kappa^{12} \) is complete by columns of \( M(e^1) \), and \( M(e^2\phi^*e^1) \subset M(e^2) \).

**Proof.** Apply Theorems 1 and 2.

3. **Biorthogonal matrices.** By considering \( M(e^1_0\kappa^*e^2) \) and \( M(e^2_0\cap e^2) \) as subsets of \( M(e^1_0) \) and \( M(e^2_0) \) respectively, we have established the one-to-one correspondence between those two subsets. The correspondences are orthogonal. But the Fourier coefficient function of every vector in \( M(e^1_0\kappa^*e^2) \) is also modular as to \( e^2 \). This property gives rise to another direction of studying the correspondences between the aforementioned subsets.

**Definition 2.** Suppose that \( e_0^1, e_1^1 \) are idempotent as to \( e^1 \), \( \kappa^{12} \) is by columns of \( M(e^1_0) \), and \( \phi^{21} \) is by rows-conjugate of \( M(e^1) \). Then \( \kappa^{12}, \phi^{21} \) are said to be biorthogonal as to \( e^1e_0^1e^2 \) in case

\[
J^1\mu^1 = J^2(J^1\mu^{12}, J^1\phi^{21}v^1)
\]

for every pair of vectors \( \mu^1, v^1 \) in \( M(e^1_0\kappa^*e^2) \), \( M(e^1_0\phi^*e^2) \) respectively.

It is obvious that \( \kappa^{12}, \phi^{21} \) are biorthogonal as to \( e^1e_0^1e^2 \) if and only if \( \phi^{*12}, \kappa^{*21} \) are biorthogonal as to \( e^1_0e^2 \).

**Definition 3.** Suppose that \( e_0^1 \) is idempotent as to \( e^1 \) and \( \kappa^{12}, \phi^{*12} \) are by columns of \( M(e^1_0) \). Then \( \kappa^{12}, \phi^{21} \) are said to be biorthogonal as to \( e^1e_0^1e^2 \) if they are biorthogonal as to \( e^1_0_0e^2 \).

**Theorem 4.** Suppose that \( e_0^1, e_1^1 \) are idempotent as to \( e^1 \), and \( \kappa^{12}, \phi^{*12} \) are by columns of \( M(e^1_0), M(e^1_1) \) respectively. Then

(i) \( \kappa^{12}, \phi^{21} \) are biorthogonal as to \( e^1\phi^1e^2 \) if \( \kappa^{12}, \phi^{21} \) are biorthogonal as to \( e^1_0_0e^2 \), and only if \( \kappa^{12}, \phi^{21} \) are biorthogonal as to \( e^1_0e_1^1e^2 \);

(ii) \( \kappa^{12}, \phi^{21} \) are biorthogonal as to \( e^1_0_0e^2 \) if and only if

\[
J^2(J^1\mu^{12}, \mu^2) = J^2(J^1\mu^{12}, \mu^2)
\]

holds for every \( \mu^1 \) in \( M(e^1_0\phi^*e^2) \) and every \( \mu^2 \) in \( M(e^2_0\cap e^2) \).

(iii) \( \kappa^{12}, \phi^{21} \) are biorthogonal as to \( e^1\phi^1e^2 \) if and only if

\( e_0^1 \) is idempotent as to \( e^1 \) in case \( e_0^1 \) is by columns of \( M(e^1) \) and \( J^1e_0^1 = e_0^1 \). See G.A., I, pp. 23–24.
(4.2) $J^{2\xi^2_2\eta^2} = J^1(J^{2\phi^*2\phi^*21}, J^{2\kappa^{12}\eta^2})$

holds for every $\xi^2$ in $\mathcal{M}(\varepsilon^2 \cap \varepsilon_2^*)$ and every $\eta^2$ in $\mathcal{M}(\varepsilon^2 \cap \varepsilon_2^*)$.

**Proof.** Since $\mathcal{M}(\varepsilon^1_0)$ and $\mathcal{M}(\varepsilon^1_1)$ are subsets of $\mathcal{M}(\varepsilon^1_0)$ and $\mathcal{M}(\varepsilon^1_1)$ respectively, we have the conclusion (i).

To prove the necessity in (ii), consider any $\mu^2$ in the set $\mathcal{M}(\varepsilon^2_1 \cap \varepsilon_2^*)$ and any vector $\mu^1$ in $\mathcal{M}(\varepsilon^1_0 \varepsilon_1^2)$. Theorem (D) shows the existence of a vector $\nu^2$ in $\mathcal{M}(\varepsilon^1_0 \kappa^* \varepsilon^2)$ such that

(a) $\nu^1 = J^{2\kappa_{12}} \mu^2, \quad \mu^2 = J^{1 \kappa_{*21} \nu^1}$.

By using the fact that $\kappa^{12}$ is modular as to $\varepsilon^1_2 \varepsilon^2_2$, we have

(b) $J^{2\varepsilon(J^1 \mu^{12}, \mu^2)} = J^1 \mu^1 \nu^1$.

Equations (a) and (b) show that the condition is necessary. The sufficiency can be proved similarly.

Now observe that $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\varepsilon^1_2 \varepsilon^2_2$ if and only if

(c) $J^{2}(J^1\mu^{12}, \eta^2) = J^1(\mu^1, J^{2\kappa^{12} \eta^2})$

for every $\mu^1$ in $\mathcal{M}(\varepsilon^1_0 \phi^* \varepsilon^2)$ and every $\eta^2$ in $\mathcal{M}(\varepsilon^2_1 \cap \varepsilon_2^*)$. The latter condition is obviously satisfied if and only if (4.2) holds, since $\mathcal{M}(\varepsilon^1_0 \phi^* \varepsilon^2)$ and $\mathcal{M}(\varepsilon^2_1 \cap \varepsilon^2)$ are in one-to-one correspondence via the transformations $J\phi^{21}$ and $J^{2\phi^*21}$ by (D).

**Theorem 5.** Suppose that $\varepsilon^1_0$, $\varepsilon^1_1$, $\kappa^{12}$, $\phi^{21}$ have the same properties as in the hypothesis of Theorem 4, and $\mathcal{M}(\varepsilon^1_0)$ contains either $\mathcal{M}(\varepsilon^1_0)$ or $\mathcal{M}(\varepsilon^1_1)$. Then $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\varepsilon^1_2 \varepsilon^1_2 \varepsilon^2$ if and only if $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\varepsilon^1_2 \varepsilon^2_2$.

**Proof.** The condition is necessary by Theorem 4. If $\mathcal{M}(\varepsilon^1_0)$ contains $\mathcal{M}(\varepsilon^1_0)$, the condition is obviously sufficient. If $\mathcal{M}(\varepsilon^1_0)$ contains $\mathcal{M}(\varepsilon^1_1)$, the sufficiency is proved by the following argument: Let $\mu^1$, $\nu^1$ be vectors in $\mathcal{M}(\varepsilon^1_1 \phi^* \varepsilon^2)$, $\mathcal{M}(\varepsilon^1_0 \kappa^* \varepsilon^2)$ respectively. Since $\mu^1$ is modular as to $\varepsilon^1_0$ by hypothesis, we have

$$J^{1 \mu^1 \nu^1} = J^1(J^{1 \mu^1 \varepsilon^1_0}, \nu^1) = J^1(\mu^1, J^{1 \varepsilon^1_0 \nu^1}).$$

Also

$$J^{1 \kappa^{*21} \nu^1} = J^1(J^{1 \kappa^{*21} \varepsilon^1}, \nu^1) = J^1(\kappa^{*21}, J^{1 \varepsilon^1 \nu^1}).$$

The vector $J^{1 \varepsilon^1_0 \nu^1}$ is evidently in the set $\mathcal{M}(\varepsilon^1_0 \kappa^* \varepsilon^2)$.

**Corollary 6.** Suppose that $\kappa^{12}$, $\phi^{*12}$ are by columns of $\mathcal{M}(\varepsilon^1)$, $\mathcal{M}(\varepsilon^1)$
respectively. Then $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1\epsilon_{1}^{2} e^2$ if and only if $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1 e_{1}^{2} e^2$.

**Theorem 7.** Suppose that $\epsilon_1, \epsilon_2$, $\kappa^{12}$, $\phi^{21}$ have the same properties as in the hypothesis of Theorem 4, and $M(\epsilon_2^2 \cap \epsilon^2)$ is everywhere dense in $M(\epsilon_2)$. Then $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1\epsilon_{1}^{2} e^2$ and the orthogonal complement within $M(\epsilon_2)$ of $M(\epsilon_1)$ is a subset of $M(\epsilon_0)$ if and only if $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1 e_{1}^{2} e^2$ and $\phi^{21}$ is complete by rows-conjugate of $M(\epsilon_1)$.

**Proof.** The condition is obviously sufficient. For the necessity, we need to show only that $\phi^{21}$ is complete by rows-conjugate of $M(\epsilon_1)$. Consider any $\xi^1$ modular as to $\epsilon_1$ such that $J^1\phi^{21} = J^1\phi^{21} = 0$. Then $\xi^1$ belongs to $M(\epsilon_1\phi^2)$. Hence

$$J^{1}\mu^1 \xi^1 = J^{2}(J^{1}\mu^{1}\kappa^{12}, J^{1}\phi^{21}\xi^1) = 0$$

for every $\mu^1$ belonging to $M(\epsilon_1^1\phi^2)$. Now if $M(\epsilon_2^2 \cap \epsilon^2)$ is everywhere dense in $M(\epsilon_2)$, then $M(\epsilon_1^1\phi^2)$ is everywhere dense in $M(\epsilon_0)$, or, $O^1M(\epsilon_1^1\phi^2) = O^1M(\epsilon_0)$. Since $\xi^1$ is in $M(\epsilon_0)$ and also in $O^1M(\epsilon_0)$, it follows that $\xi^1 = 0$.

**Theorem 8.** Assume that $\kappa^{12}$ is of type $M(\epsilon_1)\overline{M}(\epsilon^2)$, and $\phi^{21}$ is by rows-conjugate of $M(\epsilon_1)$ such that $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1 e_{1}^{2} e^2$. Then

1. $\phi^{12}$, $\kappa^{21}$ are contraceding as to $e^1 e_{1}^{2} e^2$;
2. $M(\epsilon_2^2 \cap e_{1}^{2}) \subseteq M(e_{1}^{2}\kappa_{1}^2)$;
3. $\mu^1 = J^{2}\kappa^{12} J^{1}\phi^{21} \mu^1$ for every $\mu^1$ in $M(\epsilon_1^1\phi^2)$;
4. $M(\epsilon_2^2 \cap e_{1}^{2}) \subseteq \{ J^{2}\phi^{21} \mu^1 | \mu^1 \text{ in } M(\epsilon_1^1 \cap e_{1}^{2}) \}$;
5. $M(\epsilon_1^1\phi^2) \supset M(\epsilon_1^1 \cap e_{1}^{2})$ if and only if $M(e_{1}^{2}\kappa_{1}^2)$ contains $\{ J^{2}\phi^{21} \mu^1 | \mu^1 \text{ in } M(\epsilon_1^1 \cap e_{1}^{2}) \}$.

**Proof.** Since $\kappa^{12}$ is by rows-conjugate of $M(\epsilon_1^2 \cap e_{1}^{2})$, we make use of (4.2) in Theorem 4 and secure

$$J^{2}\kappa^{12} \mu^2 = J J^{1}\epsilon_{1}^{2} e_{1}^{2} e^2 \phi^{12} \mu^2 = J^{2} \phi^{21} \mu^2$$

for every $\mu^2$ belonging to $M(\epsilon_2^2 \cap e_{1}^{2})$. This proves (1). Part (2) is an immediate consequence of (1) just proved. For the demonstration of (3), we note that $\epsilon_{1}^{2}$ is by columns of $M(\epsilon_1^1\phi^2 e_{1}^{2})$ and hence by Definition 2, we have

$$J^{1}\epsilon_{1}^{2} \eta^1 = J^{2}(J^{1}\epsilon_{1}^{2} \eta^1, J^{1}\epsilon_{1}^{2} \eta^1) = J^{2} \kappa^{12} J^{1} \phi \eta^1$$

for every $\eta^1$ in $M(\epsilon_1^1\phi^2 e_{1}^{2})$. Since $M(\epsilon_1^1\phi^2 e_{1}^{2})$ is a subset of $M(\epsilon_1^1)$, we have $J^{1}\epsilon_{1}^{2} \eta^1 = \eta^1$, and hence, part (3). To prove (4), consider any $\mu^2$ in
$\mathcal{M}(e^2 \cap e^2_\star)$. Let $\mu^1$ be a vector in $\mathcal{M}(e^1_\star \cap e^2_\star)$ for which $J^{2\phi} \phi^{*12} \mu^2 = \mu^1$. Then by (1)

$$J^1 \phi^1 \mu^1 = J^1 \phi^1 J^{2\phi} \phi^{*12} 2 \mu^2 = J^{2\phi} \phi^{*} \mu^2 = \mu^2,$$

whence (4) follows. In part (5), if $\mathcal{M}(e^1_\star \phi e^2) \supset \mathcal{M}(e^1_\star \cap e^2_\star)$, then

$$[J^1 \phi^1 1 \mu^1 | 1 \mu^1 \text{ in } \mathcal{M}(e^1_\star \phi e^2_\star)] \supset [J^1 \phi^1 1 \mu^1 | 1 \mu^1 \text{ in } \mathcal{M}(e^1_\star \cap e^2_\star)].$$

By (C), the left-hand side is identical with $\mathcal{M}(e^2 \cap e^2_\star)$, which by part (2) is a subset of $\mathcal{M}(e^2_\star \phi e^2)$. The converse is obvious.

**Theorem 9.** Let $e^1_\star$ be idempotent as to $e^1$, and $\kappa^{12}$ be of type $\mathcal{M}(e^1_\star) \bar{\mathcal{M}}(e^2)$. Suppose that $\phi^{21}$ is by rows-conjugate of $\mathcal{M}(e^1_\star)$ and $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1_\star e^2$. Then we have the following conclusions:

1. $\phi^{21}$ is complete by rows-conjugate of $\mathcal{M}(e^1_\star)$;
2. $\mu^1 = J^2 \kappa^{12} J^1 \phi^{21} 1 \mu^1$ for every $\mu^1$ in $\mathcal{M}(e^1_\star \phi e^2)$;
3. $\mathcal{M}(e^1_\star \phi e^2) = [J^2 \kappa^{12} \mu^2 | 2 \mu^2 \text{ in } \mathcal{M}(e^1_\star \cap e^2_\star)] \subset \mathcal{M}(e^1_\star \cap e^2_\star)$;
4. $\mathcal{M}(e^2 \cap e^2_\star) \subset \mathcal{M}(e^2_\star \kappa e^1)$;
5. if $\phi^{21}$ is by columns of $\mathcal{M}(e^2)$, then $e^1_\star = J^2 \kappa^{12} \phi^{21}$ and $\kappa^{12}$ is complete by columns of $\mathcal{M}(e^1_\star)$;
6. $\mathcal{M}(e^1_\star \phi e^2) \supset \mathcal{M}(e^1_\star \cap e^2_\star)$ if and only if every vector in $\mathcal{M}(e^1_\star \cap e^2_\star)$ is expressible in the form $J^2 \kappa^{12} \mu^2$, where $\mu^2$ is a vector in $\mathcal{M}(e^2 \cap e^2_\star)$.

**Proof.** Part (1) follows from Theorem 7, for $\mathcal{M}(e^2 \cap e^2_\star)$ is everywhere dense in $\mathcal{M}(e^2_\star)$ when $\kappa^{12}$ is of type $\mathcal{M}(e^1_\star) \bar{\mathcal{M}}(e^2)$. Part (2) is proved in the same way as part (3) of Theorem 8 with the replacement of $e^1$ by $e^1_\star$. By Theorem (C), part (3) is a consequence of (2). Part (4) follows from (3). For part (5), we note that $e^1_\star$ is by columns of $\mathcal{M}(e^1_\star \phi e^2)$; hence the first conclusion follows from (2) whereas the second follows from (1) and the fact that $\phi^{*12}$, $\kappa^{*21}$ are biorthogonal as to $e^1_\star e^2$. Part (6) is a consequence of (3).

**Theorem 10.** Suppose that $\kappa^{12}$ is of type $\mathcal{M}(e^1) \bar{\mathcal{M}}(e^2)$, $\phi^{21}$ is by rows-conjugate of $\mathcal{M}(e^1_\star)$, and $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $e^1_\star e^2$. Then the following assertions are equivalent:

1. $\mathcal{M}(e^2 \cap e^2_\star) \supset \mathcal{M}(e^2_\star e^1)$;
2. $\kappa^{12}$ is complete by rows-conjugate of $\mathcal{M}(e^2)$ and $\mathcal{M}(e^1_\star \phi e^2)$ contains $\mathcal{M}(e^1_\star \cap e^2_\star)$;
3. $\kappa^{12}$ is complete by rows-conjugate of $\mathcal{M}(e^2)$ and $\mathcal{M}(e^2_\star e^1)$ contains $[J^1 \phi^{21} 1 \mu^1 | 1 \mu^1 \text{ in } \mathcal{M}(e^1_\star \cap e^2_\star)]$;
4. $\kappa^{*21}$, $\phi^{*12}$ are contraceding as to $e^2_\star e^1$, and $\kappa^{12}$ is complete by rows-conjugate of $\mathcal{M}(e^2)$;
If one of the preceding five conditions is valid, then $\varepsilon^2 = \varepsilon^2\star = J^1\phi^{21}\kappa^{12}$.

**Proof.** The equivalence of the second and third conditions follows from (5) of Theorem 8. Conditions (1), (2), (4), and (5) are equivalent because of Theorem 2, where $\kappa, \phi, \varepsilon^1, \varepsilon^2, \varepsilon^3$ are replaced by $\kappa\star, \phi\star, \varepsilon^1, \varepsilon^2, \varepsilon^3$, respectively. The relation $\varepsilon^2 = J^1\phi^{21}\kappa^{12}$ follows from (4), since $\kappa^{12}$ is by columns of $M(e^1_1 \land e^1_2)$. 

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