The following theorem has been recently proved by H. L. Garabedian: If $a_n$ is a polynomial in $n$ of degree $k - 1$, the series $\sum_{n=0}^{\infty} ( -1)^n a_n$ is summable $(C, k)$ but not $(C, k - 1)$ to the value $\sum_{i=0}^{k-1} \Delta^i a_0$. Here and elsewhere in this paper $k$ is understood to be a fixed positive integer.

Our present object is to obtain some extensions of this result. One of these may be stated at once, the proof being given at the end of the paper.

**Theorem 2.** Let $a_n$ be a polynomial in $n$ of degree $k - 1$, and let $z$ be a complex number such that $|z| = 1, z \neq 1$. Then the series $\sum_{n=0}^{\infty} a_n z^n$ is summable $(C, k)$ but not $(C, k - 1)$ to the value $-2^{-i+1} a_0$.

Before stating our first theorem we require the following definitions. Let $f_n$ be a periodic function of the integer $n$; that is to say, let there be an integer $p$ such that $f_{n+p} = f_n$ for all values of $n$. Let $M(f)$ denote the mean value of $f_n$ over a period, thus $M(f) = \frac{1}{p} \sum_{n=0}^{p-1} f_n$. Suppose now that $f_n$ is a periodic function of mean value zero; that is, let $M(f) = 0$. Writing $f_n = a(n)$ in place of $f_n$, $a(n)$, set

$$
\begin{align*}
F_1(n) &= \sum_{j=0}^{n} f(j), \\
F_2(n) &= \sum_{j=0}^{n} F_1(j), \\
F_3(n) &= \sum_{j=0}^{n} F_2(j),
\end{align*}
$$

and so on; this procedure ensures that $F_i(n)$ is periodic with mean value zero ($i = 1, 2, \cdots$). In terms of these definitions we prove the following theorem:

**Theorem 1.** Let $a(n)$ be a polynomial in $n$ of degree $k - 1$, and $f(n)$ a periodic function with mean value zero. Then the series $\sum_{n=0}^{\infty} f(n)a(n)$ is summable $(C, k)$ but not $(C, k - 1)$ to the value $\sum_{i=0}^{k-1} \Delta^i a_0$.

If $f(n) = (-1)^n$ it is easily verified that $c_{i+1} = 2^{-i-1}$, which shows
that the present theorem includes that of Garabedian as a special case. The method of proof is entirely different.

We shall make use of the notation of finite differences. We recall that
\[
\Delta a(n) = a(n) - a(n+1), \quad \Delta^2 a(n) = \Delta(\Delta a(n)), \quad \cdots, \quad \text{and } E^ja(n) = a(n+j).
\]
We shall require the well known equation, in symbolic form,

\[
(1) \quad \Delta^k = (1 - E)^k = 1 - C_{k,1}E + C_{k,2}E^2 - \cdots + (-1)^kE^k,
\]

where the \(C_{k,i}\) are the binomial coefficients. We also write
\[
n^{(k)} = n(n-1)(n-2) \cdots (n-k+1),
\]
and by convention \(n^{(0)} = 1, \Delta^0 a(n) = a(n)\); then the successive differences of \(n^{(k)}\) are given by

\[
(2) \quad \Delta^i n^{(k)} = (-1)^i k^{(i)} n^{(k-i)}, \quad 0 \leq i \leq k.
\]

We require two algebraic identities, which are stated as lemmas.

**Lemma 1.** We have, for \(r\) a positive integer,

\[
\sum_{n=0}^{r} f(n)a(n) = \sum_{i=0}^{k-1} c_{i+1} \Delta^i a(0) + \sum_{i=0}^{k-1} F_{i+1}(r) \Delta^i a(r + 1) + \sum_{n=0}^{r} F_k(n) \Delta^k a(n).
\]

For,

\[
\sum_{n=0}^{r} f(n)a(n) = f(0)a(0) + f(1)a(1) + \cdots + f(r)a(r)
\]

\[\begin{align*}
&= f_1(0)a(0) + [f_1(1) - f_1(0)]a(1) + \cdots \\
&\quad + [f_1(r) - f_1(r-1)]a(r) \\
&= [F_1(0) + c_1]a(0) + [F_1(1) - F_1(0)a](1) + \cdots \\
&\quad + [F_1(r) - F_1(r-1)]a(r) \\
&= c_1 a(0) + F_1(0)[a(0) - a(1)] + \cdots \\
&\quad + F_1(r-1)[a(r-1) - a(r)] \\
&\quad + F_1(r)[a(r) - a(r+1)] + F_1(r)a(r+1) \\
&= c_1 a(0) + F_1(r)a(r+1) + \sum_{n=0}^{r} F_1(n) \Delta a(n).
\end{align*}
\]

The lemma follows by repeated applications of this device.

**Lemma 2.** The expressions

\[
(3) \quad \frac{(r+1-n)(r+2-n) \cdots (r+k-n)}{(r+1)(r+2) \cdots (r+k)}
\]

and
are algebraically identical.

To prove this, set

\[ G(t) = \frac{n(n + 1) \cdots (n + t - 1)}{(r + 1)(r + 2) \cdots (r + t)}, \quad t \text{ a positive integer}, \]

\[ G(0) = 1. \]

Then, if \( E G(t) = G(t+1) \), the expression (4) may be written

\[ G(0) - C_{k,1} EG(0) + C_{k,2} E^2 G(0) - \cdots + (-1)^k E^k G(0) = (1 - E)^k G(0) = \Delta^k G(0), \]

by (1). We have therefore to show that \( \Delta^k G(0) \) is equal to the expression (3). We write

\[ \Delta G(t) = \frac{n(n + 1) \cdots (n + t - 1)}{(r + 1)(r + 2) \cdots (r + t)} - \frac{n(n + 1) \cdots (n + t)}{(r + 1)(r + 2) \cdots (r + t + 1)} \]

\[ = \frac{n(n + 1) \cdots (n + t - 1)}{(r + 1)(r + 2) \cdots (r + t + 1)} (r + 1 - n), \quad t \geq 1. \]

In particular

\[ \Delta G(1) = \frac{n}{(r + 1)(r + 2)} (r + 1 - n). \]

Also, since \( G(0) = 1 \) and since \( G(1) = \frac{n}{r+1} \), we have \( \Delta G(0) = (r+1-n)/(r+1) \). Similarly we find

\[ \Delta^2 G(t) = \frac{n(n + 1) \cdots (n + t - 1)}{(r + 1)(r + 2) \cdots (r + t + 2)} (r + 1 - n)(r + 2 - n), \]

\[ \Delta^2 G(1) = \frac{n}{(r + 1)(r + 2)(r + 3)}, \quad \Delta^2 G(0) = \frac{(r + 1 - n)(r + 2 - n)}{(r + 1)(r + 2)}. \]

Proceeding in this way, we obtain the desired result by induction.
Proof of Theorem 1. The series \( \sum_{n=0}^{\infty} f(n) a(n) \) is said to converge \((C, k)\) to the value \( S \) provided the expression

\[
\sum_{n=0}^{r} \frac{(r + 1 - n)(r + 2 - n) \cdots (r + k - n)}{(r + 1)(r + 2) \cdots (r + k)} f(n) a(n)
\]
tends to \( S \) as \( r \) becomes infinite. Now any polynomial \( a(n) \) of degree \( k - 1 \) can be expressed in the form \( A_1(n-1)^{(k-1)} + A_2(n-1)^{(k-2)} + \cdots \) where \( A_1, A_2, \cdots \) are constants. Hence we need only consider

\[
\sum_{n=0}^{r} \frac{(r + 1 - n)(r + 2 - n) \cdots (r + k - n)}{(r + 1)(r + 2) \cdots (r + k)} (n - 1)^{(k-1)} f(n),
\]

which by Lemma 2 is equal to

\[
\sum_{n=0}^{r} \left\{ \frac{1}{r + 1} + C_{k,1} \frac{n}{r + 1} + C_{k,2} \frac{n(n+1)}{(r+1)(r+2)} \cdots \right\} (n - 1)^{(k-1)} f(n).
\]

We consider the terms of this expression separately. By (2) and Lemma 1 we have

\[
\sum_{n=0}^{r} (n - 1)^{(k-1)} f(n) = \sum_{i=0}^{k-1} c_{i+1} (-1)^i (k - 1)^{(i)} (n - 1)^{(k-1-i)} + \sum_{i=0}^{k-1} F_{i+1}(r) (-1)^i (k - 1)^{(i)} (n - 1)^{(k-1-i)} + \sum_{i=0}^{k-1} r(n)^{(k)} f(n),
\]

\[
\sum_{n=0}^{r} n^{(k)} f(n) = \sum_{i=0}^{k-1} F_{i+1}(r) (-1)^i (k - 1)^{(i)} (n + 1)^{(k-1)} + \sum_{i=0}^{k-1} F_{i+1}(r) (-1)^i (k - 1)^{(i)} (n + 2)^{(k+1-1-i)} + \sum_{n=0}^{r} (n + 1)^{(k+1)} f(n),
\]

\[
\sum_{n=0}^{r} (n + k - 1)^{(2k-1)} f(n) = \sum_{i=0}^{k-1} F_{i+1}(r) (-1)^i (2k - 1)^{(i)} (n + k)^{(2k-1-i)} + \sum_{n=0}^{r} (n + k - 1)^{(k-1)} f(n).
\]
Multiply the above equations by
\[ 1, \quad \frac{C_{k,1}}{r + 1}, \quad \frac{C_{k,2}}{(r + 1)(r + 2)}, \ldots, \]
respectively, and add. We then have on the left the expression (5) and on the right
\[ \sum_{i=0}^{k-1} c_{i+1} (-1)^i (k - 1)^i (-1)^{(k-1-i)} + \sum_{r=0}^{k-1} F_{i+1}(r)(-1)^i r^{(k-1-i)}(1 - E)^k(k - 1)^i \]
\[ + \text{terms involving } \sum_{n=0}^r. \]

Now, by (1), \((1 - E)^k(k - 1)^i = \Delta^k(k - 1)^i = 0\) for \(0 \leq i \leq k - 1\). Also, the terms involving \(\sum_{n=0}^r\) can easily be shown to approach zero as \(r\) becomes infinite. We have therefore proved that the series \(\sum_{n=0}^r (n-1)^{(k-1)} f(n)\) converges \((C, k)\) to the value \(\sum_{i=0}^{k-1} c_{i+1} (-1)^i (-1)^{(k-1-i)}\), or, setting \(b(n) = (n-1)^{(k-1)}\), \(\sum_{n=0}^r b(n) f(n)\) converges \((C, k)\) to the value \(\sum_{i=0}^{k-1} c_{i+1} \Delta^i b(0)\).

Theorem 1 now follows without difficulty. It is easily seen on examining the argument that no order of summation less than \(k\) will serve.

We now prove Theorem 2. If \(am\) is a rational multiple of \(\pi\), say \(am = \frac{2\pi q}{p}\), where \(p\) and \(q\) are integers, the function \(f(n) = z^n\) is periodic with period \(p\) and mean value zero. It is easily verified that, for \(m \geq 0\), the equations (a) become
\[ f_{m+1}(n) = \frac{z^m}{(z - 1)^m}, \quad c_{m+1} = -\frac{z^m}{(z - 1)^{m+1}}, \]
\[ F_{m+1}(n) = \frac{z^{m+1}}{(z - 1)^{m+1}} z^n, \]
whence Theorem 2 follows at once, as a special case of Theorem 1. If \(am\) is not a rational multiple of \(\pi\), then we define \(M(f)\) as \(\lim_{r \to \infty} (1/r) \sum_{n=0}^{r-1} f(n)\). Then the equations (b) still hold, and the theorem follows by the same argument as Theorem 1.