A CHARACTERIZATION OF THE GROUP OF
HOMOGRAPHIC TRANSFORMATIONS

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1. Introduction. The objectives of this note are three-fold: (1) to present a new differential geometric characterization of the group of homographic transformations of a complex variable, (2) to interpret in geometrical language the significance of the invariance of the Schwarzian derivative under a homographic transformation, and (3) to characterize a general homographic transformation by its unique association with two families of concentric circles.

2. Preliminaries. Let the equation

\[ w = w(z) \]

denote a conformal representation of the points \( z = x + iy \) of a region \( R \) of the \( z \)-plane on the points \( w = u + iv \) of a region \( R \) of the \( w \)-plane, whereby a general curve \( C \) is transformed into a curve \( \overline{C} \). Let \( \gamma \) and \( \overline{\gamma} \) denote the curvatures of \( C \) and \( \overline{C} \) at corresponding points \( z \) and \( w \), and let \( s \) and \( \overline{s} \) denote corresponding lengths of arc of \( C \) and \( \overline{C} \). For a given transformation (1) it is well known that the rate of variation \( ds/d\overline{s} \) is a function \( \lambda(x, y) \) which may be expressed in any one of the following forms \((u_x^2 + u_y^2)^{-1/2}, (u_x^2 + v_x^2)^{-1/2}, (u_y^2 + v_y^2)^{-1/2}, (v_x^2 + v_y^2)^{-1/2}\).

Comenetz\(^1\) (using a different notation) has obtained, by elementary methods, the formula

\[ \overline{\gamma} = \gamma \lambda + \lambda_y \cos \theta - \lambda_x \sin \theta, \]

wherein \( \theta = \arctan (dy/dx) \), which is the law of transformation of curvature in conformal mapping, and the formula

\[ d\overline{\gamma}/ds = \lambda^2 d\gamma/d\overline{s} + \lambda [\lambda_{xy} \cos 2\theta + \frac{1}{2} (\lambda_{yy} - \lambda_{xx}) \sin 2\theta], \]

which is the law of transformation of the rate of change of curvature with respect to arc length, under conformal mapping.

3. A new characterization of the group of homographic transformations. It is known that the most general directly conformal transformation which carries circles into circles (including straight lines) is a homographic transformation. The transformation (1) will be such a transformation if and only if

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For if these relations hold, it follows from (3) that
\[
\frac{d\gamma}{ds} = \lambda^2 \frac{d\gamma}{ds}.
\]
Hence circles are transformed into circles. Conversely, if circles are transformed by (1) into circles, we must have at corresponding points of an arbitrarily selected circle \(C\) and its correspondent \(\overline{C}\)
\[
\frac{d\gamma}{ds} = \lambda^2 \frac{d\gamma}{ds} = 0.
\]
Equation (3) also holds. Hence, no matter what the direction of \(C\) at \(z\) may be, the following equation must hold:
\[
\lambda(\lambda_{yy} - \lambda_{xx})(\sin 2\theta)/2 + \lambda\lambda_{xy} \cos 2\theta = 0.
\]
This equation must, therefore, be satisfied independently of \(\theta\), and conditions (4) necessarily follow. We may state, therefore, the following theorem.

**Theorem 1.** A necessary and sufficient condition that a conformal transformation be a homographic transformation is that the associated function \(\lambda(x, y)\) satisfy both of the following identities
\[
\lambda_{xx} = \lambda_{yy}, \quad \lambda_{xy} = 0.
\]
A geometric interpretation of this condition is that at a general point \(w\) of the curve \(\overline{C}\) the rate of variation \(d\gamma/ds\) of the curvature of \(\overline{C}\) per unit length of arc \(\overline{s}\) is independent of the direction of the curve \(C\) at the corresponding point \(z\).

Equation (3) shows that this is equivalent to the following characterization.

**Theorem 2.** The group of homographic transformations consists of all of the conformal transformations under which the differential form \(dyds\) is absolutely invariant.

The following theorem may be deduced, similarly, in consideration of equation (2).

**Theorem 3.** A necessary and sufficient condition that a transformation (1) be a directly conformal collineation is that the associated function \(\lambda(x, y)\) satisfy both of the identities
\[
\lambda_x = 0, \quad \lambda_y = 0.
\]
Under this condition the differential form \(\gamma ds\) is an absolute invariant of the transformation (1).
Since \( \gamma = \frac{d\theta}{ds} \) where \( dz/ds = e^{i\theta} \), the differential form \( d\gamma ds \) may be written in the form

\[
(6) \quad (d^2\theta/ds^2)(ds)^2
\]

wherein \( \theta = -i \log (dz/ds) \).

4. The Schwarzian derivative. Consider a curve defined by

\[
z = x(t) + iy(t),
\]

wherein \( x \) and \( y \) are functions of a real variable \( t \). It is known that the Schwarzian derivative

\[
\{z, t\} = \frac{d^2z/dt^2}{dz/dt} - \frac{1}{2} \left[ \left( \frac{d^2z/dt^2}{dz/dt} \right)^2 \right]
\]

is an absolute invariant under the homographic transformations. Let us investigate the geometric significance of the invariance of this derivative.

We find that

\[
(7) \quad \frac{z'''}{z'} = i\gamma
\]

where accents indicate differentiation with respect to \( s \). On differentiating the members of equation (7) with respect to \( s \) we obtain

\[
(8) \quad \frac{z'''}{z'} - \left( \frac{z''}{z'} \right)^2 = i\gamma ds.
\]

Making use of (7) and (8) we deduce

\[
(9) \quad \{z, s\} = i\gamma ds + \gamma^2/2.
\]

If we make a change of variable by the formula\(^2\)

\[
(10) \quad \{z, t\} = \{z, s\}(ds/dt)^2 + \{s, t\},
\]

we obtain

\[
(11) \quad \{z, t\} = (i\gamma ds + \gamma^2/2)(ds/dt)^2 + \{s, t\}.
\]

The real and imaginary components of \( \{z, t\} \),

\[
R = (\gamma^2/2)(ds/dt)^2 + \{s, t\}, \quad I = (d\gamma/ds)(ds/dt)^2,
\]

are, themselves, absolute invariants of the group of homographic transformations. Thus, if (1) is homographic, we have

\[
(12) \quad (\gamma^2/2)(ds/dt)^2 + \{s, t\} = (\gamma^2/2)(d\bar{s}/d\bar{t})^2 + \{\bar{s}, \bar{t}\},
\]

\[
(13) \quad (d\gamma/ds)(ds/dt)^2 = (d\gamma/d\bar{s})(d\bar{s}/d\bar{t})^2.
\]

\(^2\) For the change of variable formula see, for example, Ford, *Automorphic Functions*, McGraw-Hill, 1929, p. 99.
If now we put $t=s$ in (12), we find
\[
2 \{ \dd s, s \} \equiv \gamma^2 - \gamma^2 (d\dd s/ds)^2.
\]
Likewise equation (12) yields
\[
2 \{ s, \dd s \} \equiv \gamma^2 - \gamma^2 (ds/d\dd s)^2
\]
on substituting $\dd s$ for $t$.

Equation (13) expresses the invariance of the form $d\gamma ds$. Making use of this equation and equations (14) and (15), we deduce
\[
2 \{ \dd s, s \} \equiv \gamma^2 - \gamma^2 (d\dd s/d\gamma)^2,
\]
\[
2 \{ s, \dd s \} \equiv \gamma^2 - \gamma^2 (d\gamma/d\dd s)^2.
\]
These equations express the significance of the invariance of the Schwarzian derivatives $\{ s, s \}$ and $\{ z, s \}$ as intrinsic geometric relations between any pair of curves $C, \overline{C}$ which correspond under a homographic transformation. To complete the geometric interpretations of (16) and (17) let us recall the significance of the Schwarzian derivative of a real function. Consider two real functions $\sigma = \sigma(s)$ and $\tau = \tau(s)$ which are chosen to satisfy
\[
\{ \sigma, s \} = \{ \tau, s \}
\]
identically in $s$. This relation is necessary and sufficient that $\sigma(s)$ and $\tau(s)$ be connected by a homographic transformation
\[
\tilde{\sigma}(s) = \frac{a\sigma(s) + b}{c\sigma(s) + d},
\]
wherein $a, b, c, d$ are constants and $ad - bc \neq 0$. The relation (18) is also necessary and sufficient that corresponding to any set of four values $s = s_j, (j=1, 2, 3, 4)$, the cross-ratios
\[
(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \quad (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4)
\]
are identical. If as $s$ varies, points $P$ and $\overline{P}$ describe curves whose corresponding lengths of arc are defined by $\sigma = \sigma(s)$ and $\tilde{\sigma} = \tilde{\sigma}(s)$, the movements of these points will be called projectively applicable. This designation is suggested by the property that the development of these movements along a straight line produces projectively equivalent rectilinear movements.

Corresponding to a real single-valued differentiable function $\sigma = \sigma(s)$ there exists a class $\mathcal{C}_{\sigma(s)}$ of projectively applicable movements to which a movement defined by $\tilde{\sigma} = \tilde{\sigma}(s)$ will be said to belong if $\{ \tilde{\sigma}, s \} \equiv \{ \sigma, s \}$. We shall call the Schwarzian $\{ \sigma, s \}$ the absolute projective acceleration.

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of the movements of the class \( \mathcal{E}_{\pi(t)} \) or simply the absolute projective acceleration \( \{ \sigma, s \} \).

We may now state the following theorem:

**Theorem 4.** The invariance of the Schwarzian derivative \( \{ z, t \} \) yields the identities (16) and (17) which express the absolute projective accelerations \( \{ s, s \} \) and \( \{ s, s \} \) algebraically in terms of the squares of the curvatures \( \gamma \) and \( \tilde{\gamma} \) and the square of their rate of variation \( d\gamma/d\tilde{\gamma} \).

Other interesting identities may be obtained by forming various combinations of (13), (14) and (15). One of these has the surprisingly simple form

\[
\tilde{\gamma}^2/\{ s, s \} + \gamma^2/\{ \tilde{s}, s \} = 2.
\]

5. **The magnimetric circles.** Let us consider a homographic transformation

\[
w = (az + b)/(cz + d),
\]

where \( a, b, c, d \) are constants and \( ad - bc = 1 \). Since \( \lambda(x, y) = |dz/dw| \) and for (20) \( dz/dw \) is defined by \( dz/dw = (cz + d)^2 \), we may write

\[
\lambda(x, y) \equiv (cz + d)(\bar{c}\bar{z} + \bar{d}),
\]

where \( \bar{z}, \bar{c}, \bar{d} \) denote the conjugate imaginaries of \( z, c, d \). Let the value of \( \lambda \) defined at the point \( z_1 = x_1 + iy_1 \) be denoted by \( \lambda_1 \). By making use of (20), (21) and the equation

\[
z = (- dw + b)/(cw - a),
\]

for the inverse of (20), the proof of the following theorem may be supplied by the reader.

**Theorem 5.** Through a point \( z_1 \), in the \( z \)-plane (excluding \( z_1 = \infty \) and \( z_1 = -d/c \)), there passes just one circle which by (20) is magnified in all of its elements of arc length by the constant multiple \( 1/\lambda_1 \). Similarly, through the point \( w_1 \), which corresponds by (20) to the point \( z_1 \), there passes just one circle which by (22) is magnified in all of its elements of arc length by the constant multiple \( \lambda_1 \).

These circles will be called magnimetric circles of the transformations (20) and (22). The totality of the magnimetric circles in the \( z \)-plane form a family of concentric circles, a general one of which is defined by

\[
(cz + d)(\bar{c}\bar{z} + \bar{d}) = \lambda, \quad \lambda = \text{const}.
\]

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4 Loc. cit., Footnote 3, p. 3. Cartan, in considering rectilinear motion defined by \( x = x(t) \) has called the Schwarzian \( \{ x, t \} \) "l'accélération projective du mouvement."
The corresponding magnimetric circles in the \( w \)-plane form the family of concentric circles, a general one of which is defined by

\[
(cw - a)(\bar{cw} - \bar{a}) = \frac{1}{\lambda}.
\]

Let \( r(z) \) and \( \rho(w) \) denote the radii of the circles (23) and (24), respectively. We have, clearly, that

\[
r^2(z) = \frac{\lambda}{c\bar{c}}, \quad \rho^2(w) = \frac{1}{\lambda c\bar{c}}.
\]

When \( \lambda = 1 \), equations (24) and (25) represent the isometric\(^8\) circles.

We shall refer to families in the \( z \)- and \( w \)-planes as \( z \)- and \( w \)-families, respectively. By making use of equations (23) and (24) in connection with (20), the following theorem is obtained.

**Theorem 6.** There are \( \infty^2 \) transformations of the form (20) which transform a \( z \)-family of concentric circles with an arbitrarily selected center \( z_0 \) into a \( w \)-family of concentric circles with an arbitrarily selected center \( w_0 \). There are \( \infty^1 \) of these which transform a selected circle of the \( z \)-family into a selected circle of the \( w \)-family. On requiring this pair of circles to correspond, a one-to-one correspondence among the other members of the two families is established. Finally, there is just one transformation of this infinite system which transforms a selected point \( z_1 \) on any circle of the \( z \)-family into a selected point \( w_1 \) on the corresponding circle of the \( w \)-family. The circles of these \( z \)- and \( w \)-families are the magnimetric circles of the transformation, and the product of any corresponding pair of radii is equal to \( 1/|c|^2 \).

Loc. cit., Footnote 2, pp. 23–27.