ON THE MAPPING OF QUADRATIC FORMS

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The development of this paper was suggested by a theorem pro­posed by Bliss, proved by Albert, by Reid, and generalized by Hestenes and McShane. That theorem had to do with two quadratic forms \( P(z) \) and \( Q(z) \) in real variables \( z_1, z_2, \ldots, z^n \) with real coefficients, and may be stated as follows:

If \( P(z) \) is positive at each point \( z \neq (0) \) at which \( Q(z) = 0 \), then there is a real number \( \mu \) such that the quadratic form \( P(z) + \mu Q(z) \) is positive definite.

If one considers the set of points \( M \) in the xy-plane into which the \( z^- \)space is mapped by the transformation

\[
(1) \quad x = P(z), \quad y = Q(z),
\]

he will note that the above theorem may be interpreted as asserting the existence of a supporting line of the map \( M \) which has contact with \( M \) only at \((x, y) = (0, 0)\). This suggests that the theorem is related to the theory of convex sets.

In the present paper it is proven (Theorem 1) that \( M \) is a convex set. Furthermore it is proven (Theorem 2) that if \( P(z) \) and \( Q(z) \) have no common zero except \( z = (0) \), then \( M \) is closed, and is either the entire xy-plane or an angular sector of angle less than \( \pi \). Immediate corollaries include not only the theorem quoted above, but also state­ments of criteria for the existence of (1) semi-definite, and (2) definite linear combinations \( \lambda P(z) + \mu Q(z) \). The author hopes in a subsequent paper to obtain analogous results for the general case of \( m \) quadratic forms.

Throughout the paper it is to be understood without further state­ment that \( P(z) \) and \( Q(z) \) are quadratic forms in \( z_1, z_2, \ldots, z^n \), with real coefficients, the variables \( z^i \) being restricted to real values.

1. Convexity, and the condition for \( \lambda P(z) + \mu Q(z) \geq 0 \). We give first the following theorem.
Theorem 1. Under the transformation (1), the map \( \mathcal{M} \) of the \( z \)-space onto the \( xy \)-plane is convex.

If \( A \) is a point of the map, distinct from the origin \( O \), every point of the ray \( OA \) belongs to the map, since \( P(rz) = r^2P(z) \) and \( Q(rz) = r^2Q(z) \) for every real number \( r \). Hence, if \( A \) and \( B \) are two points collinear with \( O \), and each belongs to \( \mathcal{M} \), then all points of the line segment \( AB \) belong to \( \mathcal{M} \).

We will therefore assume that \( A(x_1, y_1) \) and \( B(x_2, y_2) \) are points of \( \mathcal{M} \), not collinear with the origin, defined by

\[ x_1 = P(z_1), \quad x_2 = P(z_2), \]
\[ y_1 = Q(z_1), \quad y_2 = Q(z_2), \quad z_i = (z_1, z_2, \ldots, z_n), \]

and attempt to show that every point on the line segment \( AB \) belongs to \( \mathcal{M} \). Without loss of generality we will further assume that

\[ x_2y_1 - x_1y_2 = k^2 > 0. \]

It will suffice to show analytically that if \( \bar{t} \) is any given number such that \( 0 < \bar{t} < 1 \), then the equations

\[ P(z) = x_1 + \bar{t}(x_2 - x_1), \quad Q(z) = y_1 + \bar{t}(y_2 - y_1) \]

admit a real simultaneous solution \( z = (z_1, z_2, \ldots, z_n) \).

In (4) we make the substitution

\[ z = \rho(z_1 \cos \theta + z_2 \sin \theta) \]
where \( \rho \) and \( \theta \) are real variables, and write the results in the form

\[ \rho^2 p(\cos \theta, \sin \theta) = x_1 + \bar{t}(x_2 - x_1), \]
\[ \rho^2 q(\cos \theta, \sin \theta) = y_1 + \bar{t}(y_2 - y_1), \]

where \( p \) and \( q \) are quadratic forms in \( \cos \theta, \sin \theta \), defined by

\[ p(\cos \theta, \sin \theta) = P(z_1 \cos \theta + z_2 \sin \theta), \]
\[ q(\cos \theta, \sin \theta) = Q(z_1 \cos \theta + z_2 \sin \theta). \]

Elimination of \( \rho^2 \) from the two equations (6) imposes upon \( \theta \) the condition

\[ y_1 \rho(\cos \theta, \sin \theta) - x_1 q(\cos \theta, \sin \theta) = \bar{t}T(\theta) \]
where

\[ T(\theta) = (y_1 - y_2)p(\cos \theta, \sin \theta) - (x_1 - x_2)q(\cos \theta, \sin \theta). \]

The function \( T(\theta) \) is a quadratic form in \( \cos \theta, \sin \theta \), which has the positive value \( k^2 \) at \( \theta = -\pi/2, \theta = 0, \) and \( \theta = \pi/2 \); as can be easily veri-
fied from (7), (2), and (9). Since it can vanish for at most two values of \( \theta \) between \(-\pi/2\) and \(\pi/2\), and must be negative between any two such values if they exist, the function \( T(\theta) \) will be positive on at least one of the two intervals \(-\pi/2 \leq \theta \leq 0\) or \(0 \leq \theta \leq \pi/2\). We will suppose, for definiteness, that it is the latter, the argument being similar in the two cases.

We define a function \( f(\theta) \) by the formula

\[
\frac{y_1 p(\cos \theta, \sin \theta) - x_1 q(\cos \theta, \sin \theta)}{T(\theta)}, \quad 0 \leq \theta \leq \pi/2,
\]

which is obviously continuous on the range indicated, and which has the further properties \( f(0) = 0 \) and \( f(\pi/2) = 1 \). Hence it takes on all values between 0 and 1, and in particular there is a value \( \bar{\theta} \) such that \( f(\bar{\theta}) = \bar{\theta} \). This \( \bar{\theta} \) is then a solution of (8).

The compatibility condition (8) being satisfied by \( \theta = \bar{\theta} \), we easily satisfy the two equations (6) by taking \( p_1 = p_2 = k^2/T(\bar{\theta}) \). And the resulting

\[
z = \bar{z} = p(z_1 \cos \bar{\theta} + z_2 \sin \bar{\theta})
\]
given by (5) provides the required solution for (4).

**Corollary.** A necessary and sufficient condition that there exist real \( \lambda, \mu \), such that for all real \( z \)

\[
\lambda P(z) + \mu Q(z) \geq 0
\]
is that there exist real \( a, b \), such that the two equations \( P(z) = a, Q(z) = b \) are inconsistent for real \( z \).

The condition is necessary, since in its absence the map \( \mathcal{M} \) is the entire \( xy \)-plane, and every line \( \lambda x + \mu y = 0 \) separates the plane into a positive half-plane and a negative half-plane, each of which contains points determined by \( x = P(z), y = Q(z) \).

However, if the point \( (a, b) \) does not belong to the map, no point on the ray from the origin to \( (a, b) \) belongs to the map. Hence the origin is a boundary point of the convex set \( \mathcal{M} \), and through this boundary point there passes a supporting line \( \lambda x + \mu y = 0 \), such that

\[
\lambda P(z) + \mu Q(z) \geq 0 \quad \text{for all real } z.
\]

2. **Closure, and the conditions for** \( \lambda P(z) + \mu Q(z) > 0 \). We now prove the following theorem.

**Theorem 2.** If \( P(z) \) and \( Q(z) \) have no common zero except \( z = 0 \), then \( \mathcal{M} \) is closed as well as convex, and is either the entire \( xy \)-plane or an angular sector of angle less than \( \pi \).
Since ℳ is convex, if it is not the entire xy-plane it lies entirely in some half-plane

\[ ax + by \geq 0, \quad a^2 + b^2 = 1. \]

We first show that, under the stated hypothesis, ℳ cannot contain both rays of the boundary line \( ax + by = 0 \). Suppose it did contain the two symmetrical points \( A(b, -a) \), \( B(-b, a) \), and more explicitly that

\[ P(z_1) = b, \quad Q(z_1) = -a, \quad P(z_2) = -b, \quad Q(z_2) = a. \]

Since either \( a \) or \( b \) is certainly different from zero, we may assume the notation so chosen that \( a > 0 \). Then \( Q(z_1) < 0 \) and \( Q(z_2) > 0 \). Hence there are, in the hyperplane defined by \( z = z_1u + z_2v \), two linearly independent points \( z_0 = z_1u_0 + z_2v_0 \), \( z'_0 = z_1u'_0 + z_2v'_0 \), such that

\[ Q(z_0) = Q(z'_0) = 0. \]

Consider now the quadratic form

\[ \phi(u, v) = aP(z_1u + z_2v) + bQ(z_1u + z_2v) \]

in the two real variables \( u, v \). It is easily verified that \( \phi \) vanishes at \((u, v) = (1, 0)\) and at \((u, v) = (0, 1)\). These, together with the dependent points \((c, 0)\) and \((0, c)\), are its only possible zeros unless it vanishes identically. It does not vanish identically, since it does not vanish at \((u_0, v_0)\) or \((u'_0, v'_0)\) in view of (11) and our hypothesis. Hence, by (10), \( \phi(u, v) > 0 \) except at \((c, 0)\) and \((0, c)\). This is clearly impossible, and the contradiction proves that the map \( ℳ \) cannot contain both points \( A(b, -a) \) and \( B(-b, a) \).

We now let \( X(x, y) \) denote any point of \( ℳ \), and consider the angle \( AOX \), where \( A \equiv A(b, -a) \) and \( O \equiv O(0, 0) \). Then \( \cos AOX = (bx - ay)/(x^2 + y^2)^{1/2} \). And as the point \( z \) varies over the unit hypersphere \( ||z|| = 1 \), \( \cos AOX \) is represented by the function

\[ \psi(z) = \frac{bP(z) - aQ(z)}{[P^2(z) + Q^2(z)]^{1/2}}, \quad ||z|| = 1. \]

In view of the hypothesis, \( \psi(z) \) is continuous on this hypersphere; and since its values are bounded below by \(-1\) and above by \(+1\), it attains a minimum value \( m \geq -1 \) and a maximum value \( M \leq 1 \). It is impossible that \( m = -1 \) and \( M = 1 \), since then the map \( ℳ \) would contain both points \( A(b, -a) \) and \( B(-b, a) \). Hence \( ℳ \) consists of a closed

\[ 6 \text{ Reference may be made to Bôcher, Introduction to Higher Algebra, p. 151, Theorem 2.} \]
sector bounded by rays $OA'$ and $OB'$ such that $\cos AOA' = M$ and $\cos AOB' = m$. And angle $A'OB' < \angle AOB = \pi$.

**Corollary 1.** Necessary and sufficient conditions that there exist real $\lambda, \mu$, such that for all real $z \neq (0)$

$$
\lambda P(z) + \mu Q(z) > 0
$$

are that: (1) there exist real $a, b$, such that the two equations $P(z) = a$, $Q(z) = b$ are inconsistent for real $z$; and (2) $P(z)$ and $Q(z)$ have no common zero except $z = (0)$.

The necessity is obvious. The sufficiency follows from Theorem 2. For if $(\lambda, \mu) \neq (0, 0)$ is a point of $\mathfrak{M}$ on the bisector of its angular sector, then (12) is satisfied.

**Corollary 2.** (Bliss-Albert theorem.) If, whenever $Q(z) = 0$ and $z \neq (0)$, $P(z) > 0$; then there exists a real number $\mu$ such that $P(z) + \mu Q(z)$ is positive definite.

The conditions of Corollary 1 are obviously satisfied with $(a, b) = (-1, 0)$. Hence there exist $\lambda, \mu$, satisfying (12). If $Q(z)$ actually vanishes for some $z \neq (0)$, $\lambda$ is necessarily positive and hence may be taken equal to 1.

If, on the contrary, $Q(z)$ is definite, then the map $\mathfrak{M}$ is a closed sector of which only the vertex $(0, 0)$ is on the $x$-axis. Hence there is a line $x + \mu y = 0$ such that $x + \mu y > 0$ for all points of $\mathfrak{M}$ except $(0, 0)$. Then $P(z) + \mu Q(z)$ is positive definite.

It is perhaps worthy of note that the two conditions of Corollary 1 are completely independent. This is shown by the following four examples.

**Example 1**, in which both (1) and (2) are satisfied:

$$
P(u, v) = u^2, \quad Q(u, v) = v^2.
$$

**Example 2**, in which (1) is satisfied but (2) is not:

$$
P(u, v) = u^2, \quad Q(u, v) = uv.
$$

**Example 3**, in which (1) is not satisfied but (2) is:

$$
P(u, v) = u^2 + 2uv, \quad Q(u, v) = 2uv + v^2.
$$

**Example 4**, in which neither (1) nor (2) is satisfied:

$$
P(u, v, w, t) = u^2 + 2uv + w^2, \quad Q(u, v, w, t) = 2uv + v^2 + wt.
$$