THE $R_\lambda$-CORRESPONDENT OF THE TANGENT TO AN ARBITRARY CURVE OF A NON-RULED SURFACE

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In a recent paper$^1$ the author defined at a general point $y$ of a non-rulled analytic surface $S$ the tangent line which he calls the $R_\lambda$-correspondent of the tangent at $y$ to a general curve $C_\lambda$ of $S$. It was proved$^2$ that (i) a curve $C_\lambda$ is a curve of Darboux if and only if at each of its points the $R_\lambda$-correspondent of the tangent to $C_\lambda$ coincides with this tangent, (ii) a curve $C_\lambda$ is a curve of Segre if and only if at each of its points the tangent to $C_\lambda$ and its $R_\lambda$-correspondent are conjugate tangents of $S$.

The primary purpose of this note is to present the following simple construction for the $R_\lambda$-correspondent: Let $\Lambda$ denote a point of $C_\lambda$ distinct from $y$, let $U$, $V$ denote, respectively, the points of intersection of the asymptotic $u$- and $v$-curves passing through $y$ with the asymptotic $v$- and $u$-curves passing through $\Lambda$, and let $W$ denote the point of intersection of the tangent plane to $S$ at $y$ with the line joining the points $U$, $V$. If $y$ is held fixed while $\Lambda$ tends toward $y$ along $C_\lambda$, the point $W$ describes a curve $C_\omega$ and, except when $C_\lambda$ is a curve of Segre or is tangent at $y$ to a curve of Segre, the limit of $W$ is the point $y$. The tangent at $y$ to $C_\omega$ is the $R_\lambda$-correspondent of the tangent to $C_\lambda$ at $y$.

The validity of this construction will be proved, and in addition the following theorem will be demonstrated:

A curve $C_\lambda$ is a curve of Segre if and only if for a general point $y$ of $C_\lambda$ the limit of $W$ as $\Lambda$ tends to $y$ along $C_\lambda$ is a point $W_0$ distinct from $y$. The point $W_0$ is the intersection of the directrix of the first kind of Wilczynski with the tangent at $y$ to the corresponding curve $C_\omega$ of Darboux.

Let the homogeneous projective coordinates $y^{(1)}, \ldots, y^{(4)}$ of a general point $y$ on a non-rulled analytic surface $S$ in ordinary space be functions of asymptotic parameters $u$, $v$. The functions $y^{(i)}$ are solutions of a system of differential equations, which can be reduced by a suitable transformation to Wilczynski's canonical form

\[(1) \quad y_{uu} + 2by_{v} + fy = 0, \quad y_{vv} + 2a'y_{u} + gy = 0.\]

The coefficients of these equations are functions of $u$, $v$ which are connected by three conditions of integrability. Moreover, the coordinates

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$^2$ Loc. cit., p. 393.
$y^{(i)}$ are not solutions of any equation of the form $Ay_{uv} + B y_u + Cy_v + Dy = 0$ whose coefficients are functions of $u, v$ not all zero. This statement implies that the point $y_{uv}$, whose coordinates are the functions $y^{(i)}_{uv}$, does not lie in the tangent plane to $S$ at $y$.

An arbitrary one-parameter family $F_\lambda$ of curves of $S$ is defined by the curvilinear differential equation $dv - \lambda du = 0$, where $\lambda$ is an arbitrary function of $u, v$. We denote by $C_\lambda$ the curve of $F_\lambda$ which passes through $y$. If $u, v$ be regarded as functions of a single parameter $t$, as $t$ varies, the point $y$, whose curvilinear coordinates are $u, v$, describes a curve of $S$. This curve will be the curve $C_\lambda$ if the functions $u = u(t), v = v(t)$ are selected such that for a general value of $t$

\begin{equation}
\lambda(u, v) = \frac{v'}{u'},
\end{equation}

where accents indicate differentiation with respect to $t$.

The curvilinear coordinates of the point $\Lambda$ are given by $u(t+\Delta t), v(t+\Delta t)$. The points $U$ and $V$ are therefore given by $u(t+\Delta t), v$ and $u, v(t+\Delta t)$. The general homogeneous coordinates of the points $U$ and $V$ are consequently functions of $t$, and may be represented by the developments

\begin{equation}
U = y + y_u u' \Delta t + (y_{uu} u'^2 + y_{uu} u'') \Delta t^2/2 + (y_{uuu} u'^3 + 3y_{uu} u'' u') \Delta t^3/6 + \cdots,
\end{equation}

\begin{equation}
V = y + y_v v' \Delta t + (y_{vv} v'^2 + y_{vv} v'') \Delta t^2/2 + (y_{vvv} v'^3 + 3y_{vv} v'' v') \Delta t^3/6 + \cdots.
\end{equation}

wherein $f_i, g_i, i = 1, 2, 3$, represent functions of $u, v$ which for our purpose do not require explicit determination.

By differentiating equations (1) we find that the coefficients of $y_{uv}$ in the expressions for $y_{uuu}, y_{vvv}, y_{uuuu}, y_{vvvv}$ are $-2b, -2a', -4b$, $-4a'$, respectively. The coefficients of $y_{uv}$ in the expressions for the homogeneous coordinates of the points $U, V$ are, therefore,

\begin{align*}
&- bu'' \Delta t^2/3 - (b_u u'^4 + 3bu'' u') \Delta t^3/6 + \cdots, \\
&- a' v'' \Delta t^2/3 - (a'_v v'^4 + 3a' v'' v') \Delta t^3/6 + \cdots,
\end{align*}

respectively. The point $W$, which is the intersection of the tangent plane to $S$ at $y$ with the line joining $U, V$ has homogeneous coordinates which may be obtained by forming a linear combination of those of $U$ and $V$ which contains no $y_{uv}$ term. Hence, such a combination is
Expanding this we obtain the expression

\[(2a'v'^3 - bu'^3)y + (a'v' + 3a'v''v' - bu' - 3bu'^2u')\Delta t\] 

(5) 

for the homogeneous coordinates of the point \(W\). If \(a'v'^3 - bu'^3 \neq 0\), the limit of \(W\) as \(\Delta t\) tends to zero is, clearly, the point \(y\). Moreover, the tangent to \(C_w\) at \(y\) has the direction defined by \(dv/du = -bu'/a'v'^2\).

This is the direction of the \(R_\lambda\)-correspondent of the tangent to \(C_\lambda\) at \(y\). This completes the proof for the general case in which \(\lambda\) is not a direction of Segre.

The curve \(C_\lambda\) is a curve of Segre if and only if at each of its points the direction defined by \(\lambda = v'/u'\) satisfies the equation \(a'v'^3 - bu'^3 = 0\).

In this case it is clear from (5) that the limit of \(W\) as \(\Delta t\) tends to zero is a point \(W_0\), distinct from \(y\), whose homogeneous coordinates are given by

\[(3a'v'^2v'' - 3bu'^2u'' + a_vv'^4 - bu'u^4)y + 2(a'v'^3u'y_u - bu'^3v'y_v)\Delta t + \text{terms of order } \Delta t^2\] 

(6) 

If we divide this expression by \(a'u'^3\), make use of the condition \(a'v'^3 = bu'^3\), and make the following substitutions, \(v'/u' = \lambda\), \(-v''u''/u'^3 = \lambda_u\), \(v''/u'v' = \lambda_v\), we obtain the simpler form

\[(3[\lambda_u + \lambda\lambda_v]/\lambda + a_v'\lambda/a' - b_u/b)y + 2(y_u - \lambda y_v), \quad \text{where } a'\lambda^3 = b,\] 

(7) 

for the coordinates of \(W_0\). It is, clearly, a simple matter to evaluate (7) explicitly for a direction \(\lambda = \epsilon(b/a')^{1/3}\), wherein \(\epsilon\) is a cube root of unity. The result is

\[2y_u - a_u'y/a' - \epsilon(b/a')^{1/3}(2y_v - b_vy/b).\] 

(8) 

This expression is a linear combination of the expressions \(2y_u - a_u'y/a'\) and \(2y_v - b_vy/b\) for the homogeneous coordinates of the points in which the directrix of the first kind intersects the asymptotic \(u\)- and \(v\)-tangents to \(S\) at \(y\). Moreover, the ratio of the coefficient of \(y_v\) to that of \(y_u\) is the direction \(-\epsilon(b/a')^{1/3}\) of Darboux which corresponds to the direction \(\epsilon(b/a')^{1/3}\) of Segre. This completes the demonstration of the theorem.

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