RATIONAL APPROXIMATIONS TO IRRATIONALS

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It is well known that if \( p/q \) is a convergent to the irrational number \( x \), then \(|x-p/q|<1/q^2\). The immediate converse is of course false but I have not seen in the literature any statement of the converse which is given below.

**Theorem 1.** If \( p \) and \( q \) are coprime, \( q>0 \), and if \(|x-p/q|<1/q^2\), then necessarily \( p/q \) is one of the three (irreducible) fractions

\[
p'/q', \quad (p' + p'')/(q' + q''), \quad (p' - p'')/(q' - q''),
\]

where \( p''/q'' \), \( p'/q' \) are two consecutive convergents to the irrational \( x \). One at least of the two fractions \((p' + \epsilon p'')/(q' + \epsilon q'')\) where \( \epsilon = \pm 1 \) satisfies the inequality.

In other words if the inequality is satisfied, then

\[
p/q = \left[ a_1, a_2, \cdots, a_{n-1}, a_n + c \right], \quad c = 0, \pm 1,
\]

where \([a_1, a_2, \cdots, a_r, \cdots] = x\) is the infinite simple continued fraction for \( x \), so that the \( a_i \) are integers, \( a_i \geq 1 \) (\( i \geq 2 \)).

Suppose that \( x-p/q = \theta/q^2 \), \( 0 < \theta < 1 \), \( \epsilon = \pm 1 \). Let

\[
p/q = \left[ b_1, b_2, \cdots, b_m \right], \quad p'/q' = \left[ b_1, b_2, \cdots, b_{m-1} \right],
\]

where \( m \) (which we can choose to be odd or even) is taken so that \(( -1)^{m-1} = \epsilon \). Defining \( y \) by the equation

\[
x = \left[ b_1, b_2, \cdots, b_m, y \right] = (yp + p')/(yq + q'),
\]

we obtain \( \epsilon \theta = q^2(x-p/q) = (p'q - pq')q/(yq + q') \); so that, since \( p'q - pq' = (-1)^{m-1} = \epsilon \), \( y + q'/q = 1/\theta \).

Since \( 1/\theta > 1 \) and \( q'/q < 1 \) it follows that \( y > 0. \)

If \( y > 1 \), then \( y = \left[ b_{m+1}, b_{m+2}, \cdots \right] (b_{m+1} \geq 1, \cdots) \), and so \( x = \left[ b_{1}, b_{2}, \cdots, b_{m}, b_{m+1}, \cdots \right] \), which, since the infinite simple continued fraction is unique, shows that \( p/q = \left[ b_{1}, \cdots, b_{m} \right] \) is the \( m \)th convergent to \( x \). If however \( y < 1 \), then \( 1/y = \left[ c, b_{m+1}, b_{m+2}, \cdots \right] \) with \( c \geq 1 \). But \( q/q' = \left[ b_{m}, b_{m-1}, \cdots, b_{2} \right] \) and therefore one of \( c \) and \( b_{m} \) must be unity for, if not, then \( 1/y > 2 \), \( q/q' > 2 \), \( y + q'/q < 1 < 1/\theta. \)

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1 Editor's note. In the meantime, R. M. Robinson has proved similar results in the Duke Mathematical Journal, vol. 7 (1940), pp. 354-359. Also the first part of Theorem 1 was observed by P. Fatou, Comptes Rendus de l'Académie des Sciences, Paris, vol. 139 (1904), pp. 1019–1021.
Hence \( x = [b_1, \ldots, b_{m-1}, b_m + c, b_{m+1}, \ldots] \) and \( b_i = a_i \) \((i \neq m)\), \( b_m + c = a_m \). Thus \( p/q = [a_1, a_2, \ldots, a_{m-1}, b_m] \) where \( b_m = 1 \) or \( a_m - 1 \). Consequently

\[
p/q = (p_{m-1} + p_{m-2})/(q_{m-1} + q_{m-2}), \quad b_m = 1,
\]
or

\[
p/q = (p_m - p_{m-1})/(q_m - q_{m-1}), \quad b_m = a_m - 1.
\]

The first part of the theorem is proved.

Now let \( p/q = (p_n + \epsilon p_{n-1})/(q_n + \epsilon q_{n-1}) \), \( \epsilon = \pm 1 \), \( p_n/q_n \) being the \( n \text{th} \) convergent to \( x = [a_1, \ldots, a_n, x'] = (x'p_n + p_{n-1})/(x'q_n + q_{n-1}) \). Then

\[
q^2 | x - p/q | = | x'\epsilon - 1 | (q_n + \epsilon q_{n-1})/(x'q_n + q_{n-1}),
\]

which, since \( |\epsilon| = 1 \), \( x' > 1 \), is less than unity if and only if

\[
\epsilon(x' - q_n/q_{n-1}) < 2.
\]

But this inequality is certainly satisfied when \( \epsilon \) has the sign opposite to the sign of \( x' - q_n/q_{n-1} \). The second part of the theorem follows.

Irreducible fractions \( p/q \) can be divided into three classes \([o/e], [e/o], [o/o]\) in which \( o \) and \( e \) denote odd and even integers respectively.

Since \( p_n q_{n-1} - p_{n-1} q_n = \pm 1 \) it is clear that consecutive convergents \( p_{n-1}/q_{n-1}, p_n/q_n \) belong to two different classes and hence that \( (p_n + \epsilon p_{n-1})/(q_n + \epsilon q_{n-1}) \) where \( \epsilon = \pm 1 \) must belong to the remaining class of irreducible fractions. It follows from Theorem 1 that for any irrational \( x \) infinitely many fractions of each class exist such that \( |x - p/q| < 1/q^2 \).

Theorem 1 in fact determines all such fractions.

This result is due to Scott\(^2\) who used the geometric properties of elliptic modular transformations. Scott also showed that the result is the best possible: for a given class and a fixed \( k \), \( 0 < k < 1 \), irrationals exist, dense everywhere on the real axis, such that the inequality \( |x - p/q| < k/q^2 \) is satisfied by only a finite number of fractions in the given class.

To prove the last statement it will be enough to show that, if \( x = [a_1, a_2, \ldots, a_n, \ldots] \) where the \( a_n \) are even integers not less than \( 2E + 1 \), where \( E > 1 \), then for every fraction of type \([o/o]\),

\[
\theta = q^2 | x - p/q | > 1 - 1/E.
\]

If \( \theta > 1 \), there is nothing to prove. If \( \theta < 1 \), it follows from our theo-

rem that \((p/q\) being irreducible)\n
\[
p = p_n + \epsilon p_{n-1}, \quad q = q_n + \epsilon q_{n-1}, \quad \epsilon = \pm 1,
\]

for the convergents to \(x\) are all \([e/o]\) or \([o/e]\). Write \(X = [a_{n+1}, a_{n+2}, \ldots]\), \(Y = [a_n, a_{n-1}, \ldots, a_2]\). Then if \(n \geq 2\),

\[
\theta = \frac{(Y + \epsilon)(X - \epsilon)}{XY + 1} = 1 - \frac{2 - \epsilon(X - Y)}{XY + 1} > 1 - \frac{2 + X + Y}{XY + 1},
\]

\[
XY + 1 - E(2 + X + Y) = (X - E)(Y - E) - E^2 - 2E + 1
\]

\[
> (E + 1)^2 - E^2 - 2E + 1 > 0,
\]

\[
\theta > 1 - 1/E.
\]

If \(n = 1\), then \(p = p_1 + 1, q = q_1 = 1, \theta = 1 - [0, a_2, \ldots] > 1 - 1/E.
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**MEASURABILITY AND DISTRIBUTIVITY IN THE THEORY OF LATTICES**

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**Introduction.** Garrett Birkhoff\(^2\) derived the following self-dual symmetric condition that a metric lattice be distributive:

\[
2[\mu(a \cup b \cup c) - \mu(a \cap b \cap c)] = \mu(a \cup b) - \mu(a \cap b) + \mu(a \cup c) - \mu(a \cap c) + \mu(b \cup c) - \mu(b \cap c).
\]

In a previous note\(^3\) the author introduced and discussed a generalization of Carathéodory's notion of measurability\(^4\) with respect to an outer measure function \(\mu\) which applies to arbitrary lattices \(L\). The \(\mu\)-measurable elements form a subset \(L(\mu)\) consisting of those elements \(a \in L\) which satisfy

\[
\mu(a \cup b) + \mu(a \cap b) = \mu(a) + \mu(b)
\]

for every \(b \in L\). Closure properties of \(L(\mu)\) were investigated. In par-

\(^1\) Presented to the Society, January 1, 1941. The author wishes to express his gratitude to the referee for his valuable suggestions and comments.

\(^2\) *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 81. We shall adopt the notation and terminology of this work and shall indicate specific references to it by B.

\(^3\) *A note on measure functions in a lattice*, this Bulletin, vol. 46 (1940), pp. 239–241. We shall indicate references to this paper by M.