A FIXED-POINT THEOREM FOR TREES

A. D. WALLACE

By a tree we mean a compact (= bicompact) Hausdorff space which is acyclic in the sense that

(i) if \( \mathcal{U} \) is a f.o.c. (= finite open covering) of a tree \( T \) then there is a f.o.c. \( \mathcal{B} \subset \mathcal{U} \) such that the nerve \( N(\mathcal{B}) \) is a combinatorial tree,

and which is locally connected in the sense that

(ii) if \( \mathcal{U} \) is a f.o.c. of \( T \) then there is a f.o.c. \( \mathcal{B} \subset \mathcal{U} \) whose vertices are connected sets.

It may be shown [3] that an acyclic continuous curve in the usual sense is a tree in our terminology. If \( g \) is a mapping which assigns to each point \( t \) of a topological space a set \( q_t \) in a topological space, then we say that \( q \) is continuous provided that for each \( t \) and each neighborhood \( U \) of \( q_t \) we can find an open set \( V \) containing \( t \) such that if \( t' \) is in \( V \) then \( q_{t'} \) is in \( U \). Our present purpose is to establish the following result:

(A) Let \( T \) be a tree and let \( q \) be a continuous point-to-set mapping which assigns to each point \( t \) a continuum \( q_t \) in \( T \). Then there is a \( t_0 \in T \) such that \( t_0 \in q_{t_0} \).

The proof (which is divided into several lemmas) uses strongly a technique introduced by H. Hopf [1]. However the present note has been made self-contained.

(A1) The intersection of two continua of \( T \) is again a continuum.

Proof. Let \( B_1, B_2 \) be two continua such that \( B_1 \cdot B_2 = C_1 + C_2 \) where the \( C_i \) are disjoint and closed. We can find disjoint open sets \( D_i \supset C_i \). Let \( t \in T - B_1 \cdot B_2 \). We can then find an open set \( V_t \) containing \( t \) which does not meet both \( B_1 \) and \( B_2 \). The sets \( D_i \) together with the sets \( V_t \) can be reduced to a f.o.c. \( \mathcal{U} \) of \( T \). Let \( \mathcal{B} \subset \mathcal{U} \) be the f.o.c. described in (i). Let \( \mathcal{B}_i \) be those vertices of \( \mathcal{B} \) on \( B_i \). It is easy to see that \( N(\mathcal{B}_i) \) is connected. If \( c_j \in C_j \) we can find a chain of 1-cells \( E_i \) in \( N(\mathcal{B}_i) \) whose first vertex contains \( c_1 \) and whose last vertex contains \( c_2 \). Now we cannot have \( E_i \subset D_1 + D_2 \) and \( E_i \) contains a vertex which is not on \( B_j \). Hence \( E_i \neq E_2 \) and so \( N(\mathcal{B}) \) is not a tree. This contradiction completes the proof.

1 Presented to the Society, May 3, 1941.
(A2) Any f.o.c. \( U \) of \( T \) contains a f.c.c. \( \mathfrak{f} \subseteq \mathfrak{U} \) so that each \( F_i \in \mathfrak{f} \) is connected and further \( N(\mathfrak{f}) \) is a combinatorial tree.

Proof. We can find a f.o.c. \( \mathfrak{B} \subseteq \mathfrak{U} \) such that \( N(\mathfrak{B}) \) is a tree. By a lemma due to Čech [5, p. 180] we can find a f.c.c. \( \mathfrak{f} \subset \mathfrak{B} \) such that \( \mathfrak{f} \) and \( \mathfrak{B} \) are combinatorially isomorphic. Let \( R_i \) be the f.o.c. \( (V_i, T - F'_i) \). Using (ii) it is easy to see that there is a f.o.c. \( \mathfrak{B} \) such that each \( W_i \) is connected and \( \mathfrak{B} \subseteq R_i \), for each \( i \). Let \( t \) be fixed. If \( W_j \) meets \( F'_t \) then so does \( W_j \) and so is contained in \( V_i \). Let \( Q_t \) be the union of all such \( W_j \). Then the closure of this set has a component-wise decomposition, say \( Q_t = F_{t1} + F_{t2} + \cdots + F_{t\alpha} \). Let \( \mathfrak{f} \) be the f.c.c. \( \{ F_{ij} \} \). It is clear that the elements of \( \mathfrak{f} \) are connected and it is not hard to show that \( \dim \mathfrak{f} \leq 1 \), that is, at most two elements of \( \mathfrak{f} \) have a non-null intersection. If we have a chain

\[ F_{i_1 j_1}, F_{i_2 j_2}, \ldots, F_{i_r j_r}, F_{i_1 j_1}, \quad r > 2, \]

such that each set meets the following but such that there are no other intersections, then the sets \( F_{i_1 j_1} \) and \( \sum_{i > 1} F_{i_1 j'_i} \) are connected and therefore by (A1) so is their meet, the set \( F_{i_1 j_1} \cdot F_{i_2 j_2} + F_{i_1 j_1} \cdot F_{i_r j_r} \). But then we would have \( F_{i_1 j_1} \cdot F_{i_2 j_2} \cdot F_{i_r j_r} \neq 0 \), a contradiction. It follows that \( N(\mathfrak{f}) \) is a tree.

(B) Let \( q \) be a mapping which assigns to each continuum \( K \) in \( T \) a continuum \( qK \) in \( T \) such that if \( K_1 \subseteq K_2 \), then \( qK_1 \subseteq qK_2 \). If \( \mathfrak{f} = \{ F_i \} \) is a f.c.c. with connected sets such that \( N(\mathfrak{f}) \) is a tree then there is an \( F_i \) for which \( F_i \cdot qF_i \neq 0 \).

Proof. Let \( N = N(\mathfrak{f}) \) and suppose that the vertices of \( N \) are \( e_i \). To each \( i \) we assign an \( i' \) so that \( F_i \) meets \( qF_i \). We then have a mapping \( e_i \rightarrow e_{i'} \) and since \( N \) is a tree it follows at once by a result due to Hopf [1, Lemma \( \gamma \)] that we can find an edge \( e_m e_n \) which is contained in the chain joining \( e_m \) to \( e_n \).\(^2\) We show that \( F_k \cdot qF_k \neq 0 \), \( k = m, n \). We have \( F_m \cdot F_n \neq 0 \) and by construction \( F_m \cdot qF_m \neq 0 \neq F_n \cdot qF_n \). Further

\[ (*) \]

is a simple chain of sets. Of course it may happen that \( F_n \) precedes \( F_m \) in (\( * \)) but this is of no importance. Let \( X \) be the union of all the sets in (\( * \)) from \( F_m \) up to and including \( F_n \). Let \( Y \) be similarly defined for the other part of (\( * \)). Then \( X \) and \( Y \) are continua with \( X \cdot Y = F_m \cdot F_n \).

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\(^2\) I am indebted to Professor S. Lefschetz for the remark that \( e_i \rightarrow e_{i'} \) generates a chain-mapping (that is, a mapping permutable with the boundary operator) if we define for the image of \( e_m e_n \) the chain joining \( e_m \) to \( e_n \). Since \( N \) is acyclic it follows at once that there is a fixed element. This may replace the result of Hopf.
Also $F_m + F_n$ is a continuum and so is $Z = qF_m + qF_n$. Clearly $Z$ meets the end-vertices of $(\ast)$. By $(A_1)$ $Z \cdot (X + Y)$ is a continuum. Hence $Z \cdot X \cdot Y$ is not null. Thus $F_m \cdot F_n \cdot (qF_m + qF_n) \neq 0$ and this completes the proof of $(B)$.

It is not hard to see that if $q$ is a mapping of the type described in $(A)$ then $q$ satisfies the conditions in $(B)$ if we define $qK = \sum qt$, $t \in K$, for each continuum $K$ of $T$. The proof is quite similar to those for analogous results concerning single-valued mappings.

We now turn to a proof of $(A)$. Suppose that no $t$ is in $qt$. We can find a neighborhood $R_t$ of $t$ so that $\overline{R}_t$ does not meet $qt$. Let $V_t = T - \overline{R}_t$. Since $qt \subset V_t$ we can find a neighborhood $S_t$ of $t$ so that $t' \in S_t$ implies $qt' \subset V_t$. Let $U_t$ be the meet of $R_t$ and $S_t$. We cover $T$ by a finite subcollection $\{U_i\} = \{U_{ti}\}$ of the sets $U_t$. We can find a refinement $\mathfrak{g}$ of $U = \{U_i\}$ which satisfies the conditions in $(B)$ in consequence of $(A_2)$. By $(B)$ we can find a set $F$ in $\mathfrak{g}$ so that $F$ meets $qF$. In other words we find a $t$ in $F$ such that $F$ meets $qt$. Now $F$ is in some $U_i$ and hence $qt$ is the corresponding $V_t$. But since $F$ does not meet the set $V_i$ it cannot meet $qt$. This contradiction completes the proof.

A continuous transformation $fM = N$ is said to be free (Hopf [1]) provided there is a continuous transformation $gMCZM$ such that $fgx = fx$ for each $x \in M$. The transformation $f$ is monotone if the set $f^{-1}y$ is connected for each $y \in N$.

(C) No continuum admits a free monotone transformation onto a tree.

Proof. Let $fM = T$ be monotone and $gMCN$ be continuous. For each $t \in T$ we set $qt = fgg^{-1}t$. It is not hard to see that $q$ is continuous and hence we may apply $(A)$. But from $t \in qt$ it follows at once that there is an $x \in M$ with $fgx = fx$.

The transformations $fMCN$ and $gMCN$ have a coincidence (Lefschetz [2]) if there is an $x \in M$ with $fx = gx$. As in (C) we may show that

(D) A monotone transformation $fM = T$ of a continuum onto a tree admits a coincidence with any continuous transformation $gMCN$.

Remarks. The result (A) is usually called the Scherrer fixed-point theorem when $q$ is single-valued and $T$ is an acyclic continuous curve. For a list of papers concerning it see Hopf [1]. Corollary (C) will be found in [3]. The result (A) was found while constructing a proof of (D). Finally (A) is analogous to a result of S. Kakutani [4] who has shown that if $S$ is an $n$-simplex and to each $s \in S$ we assign continuously a closed convex set $qs$ then there is an $s_0 \in qs_0$. 

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A FIXED-POINT THEOREM 759
BIBLIOGRAPHY


PRINCETON UNIVERSITY

ON THE DEFINITION OF CONTACT TRANSFORMATIONS

ALEXANDER OSTROWSKI

If \( z \) is a function of \( x_1, \ldots, x_n \) and \( p_\nu = \frac{\partial z}{\partial x_\nu} \), \( \nu = 1, \ldots, n \), a contact transformation in the space of \( z, x_1, \ldots, x_n \), is defined by a set of \( n + 1 \) equations

(a) \[ Z = Z(z, x_\mu, p_\mu), \quad X_\nu = X_\nu(z, x_\mu, p_\mu), \quad \nu = 1, \ldots, n, \]

such that firstly in calculating the \( n \) derivatives

\[ P_\nu = \frac{\partial Z}{\partial X_\nu}, \quad \nu = 1, \ldots, n, \]

the expressions for the \( P_\nu \) are given by a set of \( n \) equations

(b) \[ P_\nu = P_\nu(z, x_\mu, p_\mu), \quad \nu = 1, \ldots, n, \]

in which the derivatives of the \( p_\mu \) fall out; and secondly the equations (a) and (b) can be resolved with respect to \( z, x_\mu, p_\mu \):

(A) \[ z = z(Z, X_\mu, P_\mu), \quad x_\nu = x_\nu(Z, X_\mu, P_\mu), \quad \nu = 1, \ldots, n, \]

(B) \[ p_\nu = p_\nu(Z, X_\mu, P_\mu), \quad \nu = 1, \ldots, n. \]

These two postulates are equivalent with the hypothesis that the \( 2n + 1 \) equations (a), (b) form a transformation between the two spaces of the sets of \( 2n + 1 \) independent variables \((z, x_\nu, p_\nu), (Z, X_\nu, P_\nu)\) satisfying the Pfaffian condition

\[ dZ = \sum_{\nu=1}^{n} P_\nu dX_\nu = \rho \left( dz - \sum_{\nu=1}^{n} p_\nu dx_\nu \right), \quad \rho \neq 0. \]