ON 3-DIMENSIONAL MANIFOLDS

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Let \( P \) be a 3-dimensional manifold. \(^1\) Let \( Q \) be a 2-dimensional manifold imbedded in \( P \). Moreover, let \( P \) and \( Q \) admit of a permissible simplicial division \( K \), that is, a simplicial division of \( P \) such that some subcomplex of \( K \), say \( L \), is a simplicial division of \( Q \). Let \( K_i \) and \( L_i \) denote the \( i \)th normal subdivisions of \( K \) and \( L \), respectively. We define the neighborhood \( N_i \) of \( L_i \) to be the simplicial complex consisting of the simplexes of \( K_i \) that have at least one vertex in \( L_i \) together with the sides of all such simplexes. By the boundary \( B_i \) of \( N_i \) we mean the simplicial complex consisting of the simplexes of \( N_i \) that have no vertex in \( L_i \). Our purpose is to prove the following theorem.

**Theorem.** The boundary \( B_2 \) is a two-fold but not necessarily connected covering of \( Q \), and change of permissible division \( K \) replaces \( B_2 \) by a homeomorph of itself.

**Proof.** The neighborhood \( N_i \) is the sum of a set of 3-dimensional simplexes. Some of these 3-simplexes, say \( a_1, a_2, \ldots \), have exactly one vertex in \( L_i \), others, say \( b_1, b_2, \ldots \), have exactly two vertices in \( L_i \), while the remaining, say \( c_1, c_2, \ldots \), have three vertices in \( L_i \). Since \( K_1 \) is a normal subdivision of \( K \), the intersection of \( L_1 \) and \( b_i \) or \( c_i \) is a 1-simplex or 2-simplex, respectively. Let \( a_1, a_2, \) and \( a_3 \) be the intersections of \( B_2 \) and \( a_1, a_2, \) and \( a_3 \), respectively. We shall regard \( a_1, a_2, \) and \( a_3 \) as triangles with vertices on the 1-simplexes of \( a_1, a_2, \) and \( a_3 \). Also we shall regard \( a_2, \) and \( a_3 \) as squares with vertices on the 1-simplexes of \( a_2, \) and \( a_3 \).

Any 2-simplex of \( L_1 \), say \( ABC \), is incident to exactly two of the \( c_i \). Let \( c_1 = ABCM \). There is a unique 3-simplex of \( N_1 \), say \( \sigma \), that is incident to \( ABM \) and different from \( c_1 \). This \( \sigma \) is either a \( c_1 \), say \( c_2, \) or a \( b_1 \), say \( b_2 \). If \( \sigma = c_2 \), then the triangles \( \gamma_1 \) and \( \gamma_2 \) have a common side. Suppose that \( \sigma = b_2 = ABMN \). The 2-simplex \( ABN \) is incident to a unique 3-simplex of \( N_1 \), say \( \tau \), with \( \tau \neq ABMN \). This \( \tau \) is either \( c_3 \) or \( b_3 \). If \( \tau = b_3 \), there is a \( c_4 \), or \( b_4 \). Finally we must find a \( c_p = ABSD, D \) in \( L_1, S \) in \( B_1 \). We now consider \( \beta_2, \beta_3, \ldots, \) and \( \beta_{p-1} \). The sum of these squares is topologically equivalent to a square. One side of the square is coincident with a side of \( \gamma_1 \) and the opposite side coincident with a side of \( \gamma_p \).

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\(^1\) Our terminology is that of Seifert-Threlfall, *Lehrbuch der Topologie*. Manifolds are finite, while simplexes and cells are closed point sets.

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Since $K_1$ is a manifold, we can repeat the construction and associate with $ABC$ and $ABD$ a second pair of triangles in $B_2$ that are either incident along a common side or incident to opposite sides of a square. But there is not a third such configuration associated with $ABC$ and $ABD$. We repeat the construction for all pairs of adjacent 2-simplexes of $L_1$. Then to each 2-simplex of $L_1$ there correspond two triangles in $B_2$. Moreover, if two 2-simplexes of $L_1$ are incident along a side, the four corresponding triangles can be paired so that the two triangles of each pair either have a common side or are incident to opposite sides of a square.

Since $P$ and $Q$ are 3- and 2-manifolds, respectively, we can say that $Q$ is two-sided in $P$ in the neighborhood of any point of $Q$. Moreover, the two $\gamma$'s of $B_2$ that correspond to a 2-simplex of $L_1$ lie on opposite sides of $Q$ (in the neighborhood of this 2-simplex).

Consider a vertex $X$ of $L_1$ and the 2-simplexes $\Delta_i$ of $L_1$ that have $X$ as a vertex. On one side of $Q$ (in the neighborhood of $X$) there corresponds to each $\Delta_i$ a unique $\gamma_i$, and the $\gamma$'s have the same incidences as the corresponding $\Delta$'s (we say that two $\gamma$'s are incident if they are incident to opposite sides of a square). Let us denote by $R$ the points of these $\gamma$'s and the squares incident to pairs of these $\gamma$'s. Let $A$ denote the points of all $\alpha_i$'s that are in $a_i$'s incident to $X$ and on the side of $Q$ that we are considering.

We shall show that $R+A$ is a 2-cell. To do this we shall show that $R+A$ is a manifold relative to its boundary, that its boundary consists of one or more circles, and that any 1-cycle of $R+A$ bounds in $R+A$. First we observe that $B_2$ is a manifold; this fact follows from the structure of $B_2$ and the fact that $K_1$ is a manifold; the argument is elementary and we omit it. Since $R+A$ is the sum of 2-cells $\alpha, \beta,$ and $\gamma$, the set $R+A$ is a manifold relative to its boundary.

To show that this boundary of $R+A$ consists of one or more circles we shall study the incidences among the cells of $R+A$. First, let $a_i$ have $X$ as a vertex. If a 2-dimensional side of $a_i$ is not in $B_1$, this side must be a side of an $a_j$ or $b_j$. Furthermore, this $a_j$ or $b_j$ has $X$ as a vertex. Hence, any side of an $\alpha_i$ is also a side of an $\alpha_j$ or $\beta_j$ of $R+A$. Next, let $c_i$ have vertices $XABM, M$ in $B_1$. The sides of $\gamma_i$ that are in $XAM$ and $XBM$ are sides of $\gamma_j$'s or $\beta_j$'s of $R+A$. But the side of $\gamma_i$ in $ABM$ is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Finally, let $b_i$ have vertices $XAMN, A$ in $L_1$. The sides of $\beta_i$ in $XAM$ and $XAN$ are incident to sides of $\beta_j$'s or $\gamma_j$'s of $R+A$; the side of $\beta_i$ in $XMN$ is incident to an $\alpha_j$ or $\beta_j$ of $R+A$; but the side of $\beta_i$ in $AMN$ is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Examination of
the segments of the boundary of \( R+A \) shows that they fit together to form one or more circles.

We next show that if \( C \) is a 1-dimensional cycle of \( R+A \), then \( C \) bounds in \( R+A \). We shall find it convenient to replace \( A \) by a new set that will never be empty. We define \( A' \) to be \( A \) together with all vertices of \( \gamma \)'s of \( R \) that are not in the boundary of \( R+A \) and all sides of squares of \( R \) that are not sides of \( \gamma \)'s of \( R \) and not in the boundary of \( R+A \). If \( A \) is not empty, the set \( A' \) is the same as \( A \). But in any case \( A' \) is not empty, and \( R+A' \) is the same set as \( R+A \). The set \((R+A')-A'\) is homeomorphic to a 2-cell with an inner point removed because \((R+A')-A'\) can be obtained from the configuration of the 2-simplexes of \( L_1 \) that have \( X \) as a vertex by removing \( X \) and replacing some of the 1-simplexes by squares (open along one side). Hence, the cycle \( C \) is homologous in \( R+A' \) to a cycle on \( A' \), and we assume that \( C \) is on \( A' \). The set \( A' \) is part of \( b \), the boundary of the combinatorial neighborhood of \( X \) in \( K_2 \). Since \( K_2 \) is a manifold, the set \( b \) is a 2-sphere. Assume that \( C \) does not bound in \( A' \). Then \( C \) must surround a 2-simplex of \( b \) that is not in \( A' \). We easily find a 2-simplex of \( R+A' \) that is not incident along one of its sides to another 2-simplex of the manifold \( B_2 \). This contradiction proves that \( C \) bounds, and the proof that \( R+A \) is a 2-cell is complete.

Now we draw some lines on \( R+A \). If two \( \gamma \)'s have a common side, we draw a line coincident with this common side. If two \( \gamma \)'s are incident to a square, we draw a line across the square half way between the \( \gamma \)'s. All these lines are continued so that they meet at a point of \( A \). These lines give a subdivision of \( R+A \) that is combinatorially equivalent to the combinatorial neighborhood of \( X \) in \( L_1 \). The lines can be drawn for all \( R+A \) of \( B_2 \) and we get a subdivision of \( B_2 \) that is combinatorially equivalent to a two-fold but not necessarily connected covering of \( L_1 \).

A triangle of the covering is associated with a 2-simplex of \( L_1 \) and a side of \( Q \) (in the neighborhood of this simplex). Hence, a homeomorphism is determined between this covering and any covering obtained by changing the permissible division \( K \).

The theorem is not true with \( B_1 \) rather than \( B_2 \). For example, let \( Q \) be the boundary of a 3-simplex of \( K \). Then \( B_1 \) is a sphere and a point.

We can prove the following theorem in the same way but with much less effort.

**Theorem.** The above theorem is true if \( P \) and \( Q \) are replaced by 2- and 1-dimensional manifolds.

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