ON THE LEAST SOLUTION OF PELL'S EQUATION

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Let \( x_0, y_0 \) be the least positive solution of Pell's equation
\[ x^2 - dy^2 = 4, \]
where \( d \) is a positive integer, not a square, congruent to 0 or 1 (mod 4).
Let \( \epsilon = (x_0 + d^{1/2}y_0)/2 \). It was proved by Schur\(^1\) that
\[ \epsilon < d^{1/2}, \]
or, more precisely,
\[ \log \epsilon < d^{1/2}((1/2) \log d + (1/2) \log \log d + 1). \]
He deduced (1) from (2) by the property that
\[ d^{1/2}((1/2) \log d + (1/2) \log \log d + 1) < d^{1/2} \log d \]
for \( d > 244.69 \cdots \), and, for \( d \leq 244 \), (1) is established by direct computation. It is the object of the present note to establish a slightly better result that
\[ \log \epsilon < d^{1/2}((1/2) \log d + 1). \]

Thus (1) follows immediately without any calculation. The method used is that described in the preceding paper.

Let \((d \mid r)\) be Kronecker's symbol. (We extend the definition to include negative values of \( r \) by the relation \((d \mid r_1) = (d \mid r_2)\) for \( r_1 \equiv r_2 \) (mod \( d \)).

Let \( f \) denote the fundamental discriminant related to \( d \), that is,
\[ d = m^2f, \]
where \( f \) is not divisible by a square of odd prime and is either odd, or congruent to 8 or congruent to 12 (mod 16).

**Lemma 1.** For \( d > 0 \), we have
\[ \left( \frac{d}{r} \right) = \left( \frac{d}{-r} \right). \]


**Lemma 2.** We have

\(^1\) Göttingen Nachrichter, 1918, pp. 30–36.

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\[ \sum_r \left( \frac{f}{r} \right) e^{2\pi i nr/\ell} = \left( \frac{f}{n} \right)^{\ell/2}, \]

where \( r \) runs over a complete residue system, mod \( \ell \).

**Proof.** Landau, loc. cit., Theorem 215.

**Lemma 3.** We have

\[ \frac{1}{A^* + 1} \left| \sum_{a=1}^{A} \sum_{n=a}^{a} \left( \frac{f}{n} \right) \right| \leq \frac{1}{2} \left( f^{\ell/2} - \frac{A^* + 1}{f^{\ell/2}} \right), \]

where \( A^* \) is the least positive residue of \( A \), mod \( f \).

**Proof.** (See Lemma 1 of the preceding paper.) We have, by Lemma 2,

\[
\sum_{a=1}^{A} \sum_{n=a}^{a} \left( \frac{f}{n} \right) = \frac{1}{2} f^{\ell/2} \sum_{a=0}^{A} \sum_{n=a}^{a} \left( \frac{f}{n} \right) \sum_{r=1}^{1} \left( \frac{f}{r} \right) e^{2\pi i nr/\ell} \\
= \frac{1}{2} \sum_{a=0}^{A} \sum_{n=0}^{a} \sum_{r=1}^{1} \left( \frac{f}{r} \right) e^{2\pi i nr/\ell}.
\]

Then

\[
f^{\ell/2} \left| \sum_{a=1}^{A} \sum_{n=a}^{a} \left( \frac{f}{n} \right) \right| \leq \frac{1}{2} \sum_{r=1}^{f} \sum_{a=0}^{A} \sum_{n=0}^{a} e^{2\pi i nr/\ell} \\
= \frac{1}{2} \sum_{r=1}^{f} \left( \sin \left( A + 1 \right) ! r / \ell \right)^2 / \sin \pi r / \ell \\
= \frac{1}{2} \sum_{r=1}^{f} \left( \sin \left( A^* + 1 \right) ! r / \ell \right)^2 / \sin \pi r / \ell \\
= \frac{1}{2} \sum_{a=0}^{A^*} \sum_{n=0}^{a} e^{2\pi i nr/\ell} \\
\]

since

\[
\sum_{r=1}^{f} e^{2\pi i nr/\ell} = \sum_{r=1}^{f} e^{2\pi i nr/\ell} - 1 = \left\{ \begin{array}{ll} -1 & \text{if } \ell | n, \\ f - 1 & \text{if } f | n. \end{array} \right.
\]

**Lemma 4.** For any discriminant \( d > 0 \) and \( A > d^{1/2} \), we have
\[ \left| \sum_{a=1}^{A} \sum_{n=1}^{a} \left( \frac{d}{n} \right) \right| \leq \frac{1}{2} Ad^{1/2}. \]

**Proof.** It is well known that \(^2\)

\[ \left( \frac{d}{n} \right) = \left( \frac{f}{n} \right) \sum_{r \mid (m,n)} \mu(r). \]

Then

\[ \sum_{a=1}^{A} \sum_{n=1}^{a} \left( \frac{d}{n} \right) = \sum_{a=1}^{A} \sum_{n=1}^{a} \left( \frac{f}{n} \right) \sum_{r \mid (m,n)} \mu(r) \]

\[ = \sum_{r \mid m} \mu(r) \sum_{a=1}^{A} \sum_{n=1}^{a} \left( \frac{f}{n} \right) = \sum_{r \mid m} \mu(r) \sum_{a=1}^{A} \sum_{n=1}^{[a/r]} \left( \frac{f}{rn} \right) \]

Then, by Lemma 2,

\[ \left| \sum_{a=1}^{A} \sum_{n=1}^{a} \left( \frac{d}{n} \right) \right| \leq \frac{1}{2} \sum_{r \mid m} \left| \sum_{a=1}^{[a/r]} \sum_{b=1}^{\left[\frac{a}{f}\right]} \left( \frac{f}{n} \right) \right| \]

\[ \leq \frac{1}{2} \sum_{r \mid m} \left( \left( \left[ \frac{A}{r} \right] + 1 \right) f^{1/2} - \frac{1}{f^{1/2}} \left( \left[ \frac{A}{r} \right] + 1 \right)^2 \right) \]

\[ \leq \frac{1}{2} \sum_{r \mid m} \frac{A}{r} f^{1/2} \leq \frac{1}{2} A f^{1/2} m = \frac{1}{2} A d^{1/2}, \]

since we have \( f^{1/2} r < f^{1/2} m < A, \)

\[ f^{1/2} - \frac{1}{f^{1/2}} \left( \left[ \frac{A}{r} \right] + 1 \right)^2 < f^{1/2} - \frac{1}{f^{1/2}} \cdot \frac{f}{2} = 0 \]

and

\[ \sum_{r \mid m} 1 \leq m. \]

**Lemma 5.** We have

\[ \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n} < \frac{1}{2} \log d + 1. \]

\(^2\) This follows from the fact that \( \sum_{a \mid d} a(d) = 0 \) or \( 1 \) according as \( a > 1 \) or \( a = 1. \)
Proof. For \( n \geq 1 \) let
\[
S(n) = \sum_{a=1}^{n} \sum_{m=1}^{a} \left( \frac{d}{m} \right),
\]
and let \( S(0) = S(-1) = 0 \). Then we have
\[
S(n) - 2S(n - 1) + S(n - 2) = \left( \frac{d}{n} \right), \quad n \geq 1,
\]
and
\[
\sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n} = \sum_{n=1}^{\infty} \left\{ S(n) - 2S(n - 1) + S(n - 2) \right\} \frac{1}{n}
\]
\[
= \sum_{n=1}^{\infty} S(n) \left( \frac{1}{n} - \frac{2}{n + 1} + \frac{1}{n + 2} \right)
\]
\[
= \sum_{n=1}^{\infty} \frac{2S(n)}{n(n + 1)(n + 2)}.
\]

We divide the series into two parts
\[
S_1 = \sum_{n=1}^{A-1}, \quad S_2 = \sum_{n=A}^{\infty}.
\]

Since
\[
|S(n)| \leq \sum_{a=1}^{n} \sum_{m=1}^{a} 1 = \frac{n(n + 1)}{2},
\]
it follows that
\[
|S_1| \leq \sum_{n=1}^{A-1} \frac{1}{n + 2}.
\]

If \( A > \sqrt{d} \) we have by Lemma 4
\[
|S_2| < \sum_{n=A}^{\infty} \frac{nd^{1/2}}{n(n + 1)(n + 2)} = \frac{d^{1/2}}{A + 1}.
\]

Hence
\[
\left| \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n} \right| \leq \sum_{n=1}^{A-1} \frac{1}{n + 2} + \frac{d^{1/2}}{A + 1}
\]
\[
= \sum_{n=1}^{A-1} \frac{1}{n + 1} - \frac{1}{2} + \frac{1}{A} + \frac{1}{A + 1} + \frac{d^{1/2}}{A + 1}
\]
\[
\leq \log(A - 1) - \frac{1}{2} + \frac{1}{A} + \frac{d^{1/2} + 1}{A + 1}.
\]
Taking $A = \lceil d^{1/2} \rceil + 1$ we have
\[
\left| \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n} \right| \leq \log d^{1/2} - \frac{1}{2} + \frac{1}{d^{1/2}} + \frac{d^{1/2} + 1}{d^{1/2} + 1} = \frac{1}{2} \log d + \frac{1}{2} + \frac{1}{d^{1/2}} < \frac{1}{2} \log d + 1
\]
since $d \geq 5$.

**Theorem 1.** We have
\[
\log \epsilon < d^{1/2}((1/2) \log d + 1).
\]

**Proof.** It is known that the number $h(d)$ of classes of non-equivalent quadratic forms with determinant $d > 0$, is given by
\[
h(d) = \frac{d^{1/2}}{\log \epsilon} \sum_{n=1}^{\infty} \left( \frac{d}{n} \right) \frac{1}{n}.
\]
Since $h(d) \geq 1$, we have the theorem.

**Theorem 2 (Schur).** We have
\[
\log \epsilon \leq d^{1/2} \log d.
\]

**Proof.** For $d > e^2$, the theorem follows from Theorem 1. If $d < e^2$, then $d = 5$. Evidently $\epsilon = (3 + 5^{1/2})/2$ and
\[
\log \epsilon < 5^{1/2} \log 5.
\]