A GENERALIZATION OF THE POLAR REPRESENTATION OF NONSINGULAR MATRICES

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1. Introduction. If $A$ is a square matrix with elements in the complex number field, then

$$A = PU,$$

where $P$ is a positive definite hermitian matrix and $U$ is a unitary matrix.\footnote{L. Autonne, *Sur les groupes linéaires, réels et orthogonaux*, Bulletin de la Société Mathématique de France, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.} In this polar representation of the matrix $A$, as it is called, the two matrices $P$ and $U$ are unique. Since the matrix $P$ is positive definite and nonsingular, it has the same signature as the identity matrix $E$ while the unitary matrix $U$ is a conjunctive automorph of $E$. From (1) we may deduce a somewhat similar representation of $A$ in terms of a positive definite hermitian matrix and a conjunctive automorph, not of $E$, but of any nonsingular positive definite hermitian matrix.

Let $H$ be a nonsingular hermitian matrix which is positive definite, so that there exists a nonsingular matrix $Q$ satisfying

$$Q^{-1}H(Q^{-1})^* = E,$$

where $Q^*$ is the conjugate transposed of $Q$. If $B = Q^{-1}AQ$ and $B = PU$ is the polar representation of $B$, then $A = QPQ^{-1}QUQ^{-1} = DR$, where $D = QPQ^{-1}$ and $R = QUQ^{-1}$. Since $DH = QPQ^{-1}QQ^* = QPQ^*$, $DH$ is hermitian with the same signature as $H$. Further $RHR^* = QUQ^{-1}QQ^* \cdot (Q^{-1})^*U^*Q^* = QQ^* = H$. Hence we have this result as an analogue of the polar representation (1) of $A$.

**Result (1).** If $H$ is any nonsingular positive definite hermitian matrix and $A$ is a nonsingular matrix, then

$$A = DR,$$

where $DH$ is a positive definite hermitian matrix and $RHR^* = H$.

If $H = H^{-1}$, $A = DR = DHHR$ and, since $HRH(HR)^* = H^3 = H$, $A = P_1R_1$, where $P_1$ is a positive definite hermitian matrix and $R_1HR_1^* = H$. Therefore we have as a second analogue of (1) the following result.

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RESULT (2). If \( H \) is a positive definite hermitian matrix, whose square is the identity matrix, and, if \( A \) is a nonsingular matrix, then

\[
A = P_1R_1,
\]

where \( P_1 \) is a positive definite hermitian matrix and \( R_1 \) is a conjunctive automorph of \( H \). The two matrices \( P_1 \) and \( R_1 \) are unique.

It is our intention here to determine what corresponds to results (1) and (2), if \( H \) is a nonsingular hermitian matrix but not necessarily definite.

2. Generalization in the complex field. Let \( H \) be any nonsingular hermitian matrix and let \( A \) be a nonsingular matrix. If

\[
A = DR,
\]

where

\[
DH = HD^*\tag{5}
\]

and

\[
RHR^* = H,\tag{6}
\]

we shall call (4) a polar representation of \( A \) with respect to \( H \) or, for brevity an \( H \)-representation of \( A \). If (4), (5) and (6) are satisfied, \( AHA^*H^{-1} = DRIIR^*DR^{-1} = DHD^*H^{-1} = D^2 \). Further, if \( AHA*H^{-1} = D^2 \) and \( DH = HD^* \), then (6) is satisfied with \( R = D^{-1}A \). For \( D^{-1}AH(D^{-1}A)^* = D^{-1}AHA^*(D^{-1})^* = D^{-1}D^2H(D^{-1})^* = H \) by (5). Therefore we have proved this lemma.

**Lemma 1.** A necessary and sufficient condition that a matrix \( A \) have an \( H \)-representation is that there exist a matrix \( D \) such that \( DH = HD^* \) and \( AHA^*H^{-1} = D^2 \).

If \( A \) has the \( H \)-representation (4) and \( QAQ^{-1} = A_1, QDQ^{-1} = D_1, QRQ^{-1} = R_1 \) and \( QHQ^* = H_1 \), then \( A_1 = D_1R_1 \) where \( D_1H_1 = H_1D_1^* \) and \( R_1H_1R_1^* = H_1 \). Moreover \( H_1 = QHQ^* \) and \( A_1H_1A_1^* = QAHA^*Q^* \), so that the two pencils of hermitian matrices \( AHA^* - xH \) and \( A_1H_1A_1^* - xH_1 \) are conjunctive. Therefore, if \( A \) has an \( H \)-representation and if the two pencils \( AHA^* - xH \) and \( A_1H_1A_1^* - xH_1 \) are conjunctive, the matrix \( A_1 \) has an \( H_1 \)-representation. Accordingly we may suppose that the pencil \( AHA^* - xH \) is in a canonical or normal form previously determined.\(^2\) Since \( DH \) is hermitian, the elementary di-

visors of \(D-xE\) are the same as those of the pencil of hermitian matrices \(DH-xH\). Complex elementary divisors of such a pencil must therefore occur in conjugate pairs and in particular this is true of the pure imaginary elementary divisors. Hence, if \(D-xE\) has the elementary divisor \((x-ib)^r\) repeated \(s\) times, where \(b\) is real, then \(D-xE\) also has the elementary divisor \((x+ib)^r\) repeated \(s\) times and \(D^2-xE\) has the elementary divisor \((x+b^2)^r\) repeated \(2s\) times. Since the elementary divisors of \(D^2-xE\) are the same as those of \(AHA^* - xH\), the matrix \(A\) cannot have an \(H\)-representation if the pencil \(AHA^* - xH\) has an elementary divisor \((x+b^2)^r\), that is, a negative elementary divisor, repeated an odd number of times.

We proceed to determine what further conditions, if any, must be satisfied in order that \(A\) may have an \(H\)-representation. In canonical form the matrices \(H\) and \(AHA^*\) are similarly partitioned diagonal block matrices of such a nature that it is only necessary to consider the three special cases in which the pencil \(AHA^* - xH\) has

(i) only the two elementary divisors \((x-a)^n\), \((x-a)^n\), \(a\neq \bar{a}\),
(ii) the single elementary divisor \((x-b^2)^n\), \(b\) real,
(iii) only the two elementary divisors \((x+b^2)^n\), \((x+b^2)^n\), \(b\) real.

Case (i). In canonical form

\[
H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \quad \text{and} \quad AHA^*H^{-1} = \begin{pmatrix} a(E + U) & 0 \\ 0 & \bar{a}(E + U) \end{pmatrix},
\]

where \(U\) is the auxiliary unit matrix of order \(n\). Let

\[
G = (E + U)^{1/2} = E + U/2 - U^2/8 + \cdots + \alpha U^{n-1},
\]

\[
2^{n-4}(n - 3)!(n - 1)! \alpha = (-1)^n(2n - 5)!
\]

Then \(G' = (E + U')^{1/2}\). If \(a = \rho^2e^{i\delta}\), \(a^{1/2} = \delta e^{i\theta/2}\) and \(\bar{a}^{1/2} = \delta e^{-i\theta/2}\), where \(\delta = \pm 1\). Therefore, if \(D = [\alpha^{1/2}G, \bar{a}^{1/2}G']\), \(DH = HD^*\) and \(D^2 = AHA^*H^{-1}\).

The signature of \(DH\) is the same as that of \(H\) since the signature of both matrices is zero. Since \(\delta\) may have either of the values 1 or \(-1\), \(D\) is not unique.

Case (ii). In canonical form

\[
H = \epsilon T \quad \text{and} \quad AHA^* = b^2 (E + U)H,
\]

where \(\epsilon = \pm 1\) and \(T\) is the counter unit matrix, so that

\[
T^2 = E \quad \text{and} \quad TU = U'T.
\]

If \(D = bG\), where \(G\) is defined by (7), \(D^2 = AHA^*H^{-1}\) and \(DH = HD^*\) by (8). If \(n\) is even the signatures of \(DH\) and \(H\) are both zero and, if in \(DH\), \(b\) is replaced by \(-b\) the signature of the resulting matrix is
also zero. If $n$ is odd, the signature of $H$ is $\epsilon$ and that of $DH$ is $\pm \epsilon$ according as $b$ is positive or negative. Therefore, if $DH$ has the same signature as $H$ and $n$ is even, $D$ is not unique but, if $n$ is odd, $D$ is unique.

Case (iii). If the matrix $D$ exists the pencil $DH - xH$ has only the two elementary divisors $(x + ib)^n$, $(x - ib)^n$ and we may take $DH - xH$ in the canonical form

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \quad D = \begin{pmatrix} ibG & 0 \\ 0 & -ibG' \end{pmatrix},$$

where $G$ is defined by (7). Hence

$$AHA^*H^{-1} = D^2 = \begin{pmatrix} -b^2(E + U) & 0 \\ 0 & -b^2(E + U') \end{pmatrix}$$

and

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

while the matrix $H$ may be transformed without disturbing $D^2$ into

$$\begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}.$$ 

For a matrix pencil $AHA^* - xH$ with elementary divisors $(x + b^2)^n$, $(x + b^2)^n$ there are three distinct possible canonical forms in which $H$ is one of the matrices

$$\begin{pmatrix} \epsilon_1 T & 0 \\ 0 & \epsilon_2 T \end{pmatrix},$$

$\epsilon_1 = 1$, $\epsilon_2 = -1$; $\epsilon_1 = \epsilon_2 = 1$ or $\epsilon_1 = \epsilon_2 = -1$. If we call $\epsilon_1$ and $\epsilon_2$ the indices associated with the elementary divisor $(x + b^2)^n$, we see that $A$ has an $H$-representation, if and only if one index is positive and the other is negative. Further, even when $A$ does have an $H$-representation, $D$ is not unique. Since the canonical form for the pencil $AHA^* - xH$ is diagonal block, the following theorem follows immediately from Lemma 1.

**Theorem 1.** Let $H$ be any nonsingular hermitian matrix and $A$ be a nonsingular matrix. Then $A$ has an $H$-representation $A = DR$, if and only if the negative elementary divisors of the pencil $AHA^* - xH$ occur in pairs and exactly half of the indices associated with each negative elementary divisor are positive. The matrix, $D$, and therefore the matrix
$R$, is unique, if and only if all elementary divisors of the pencil are positive and of odd order.

If $H^2 = E$, $A = DHHR = SV$, where $S = DH$ and $V = HR$. Moreover $S = S^*$ and $VHV^* = H^2 = H$. Hence we have as a corollary the following.

**Corollary 1.** If $H^2 = E$, the matrix $A$ can be written in the form $A = SV$, where $S$ is hermitian with the same signature as $H$ and, $VHV^* = H$, if and only if the conditions of Theorem 1 are satisfied.

The known theorem on the uniqueness of the polar components of the matrix $A$ in (1) is a particular case of Theorem 1. For, if $H = E$, $AHA^* = AA^*$ and the elementary divisors of the pencil $AA^* - xE$ are all positive and linear.

3. **Modified representation of any nonsingular matrix.** Even if the matrix $A$ does not satisfy the conditions of Theorem 1, it is possible to find a somewhat different representation of $A$. We first reduce the pencil $AHA^* - xH$ by conjunctive transformation to

$$
\left( \begin{array}{ccc}
A_1H_1A_1^* & 0 & 0 \\
0 & A_2H_2A_2^* & 0 \\
0 & 0 & H_2
\end{array} \right) - x \left( \begin{array}{ccc}
H_1 & 0 & 0 \\
0 & H_2 & 0
\end{array} \right),
$$

where no elementary divisor of $A_1H_1A_1 - xH_1$ is negative and all elementary divisors of $A_2H_2A_2^* - xH_2$ are positive. Then $A_1 = D_1R_1$ and, since all elementary divisors of $-A_2H_2A_2^* - xH_2$ are positive, there exists a matrix $D_2$, such that $D_2^2 = -A_2H_2A_2^*H_2^{-1}$ and $D_2H_2 = H_2D_2^*$. If $R_2 = D_2^{-1}A_2$, then $R_2H_2R_2^* = D_2^{-1}A_2H_2A_2^*(D_2^*)^{-1} = D_2 \cdot (-D_2^*H_2)(D_2^*)^{-1} = -H_2$. Therefore $A_2 = D_2R_2$, where $D_2H_2$ is hermitian with the same signature as $H_2$ and $R_2H_2R_2^* = -H_2$. Let

$$
V = \left( \begin{array}{ccc}
E_1 & 0 & 0 \\
0 & -E_2 & 0
\end{array} \right) = [E_1, -E_2],
$$

where $E_i$ is the unit matrix of the same order as $H_i$. Then $V^2 = E$. If $A = [A_1, A_2]$, $H = [H_1, H_2]$, $R = [R_1, R_2]$ and $D = [D_1, D_2]$, then $A = DR$ where $DH$ is hermitian with the same signature as $H$ and $RHR^* = HV$, so that the signature of $HV$ is the same as that of $H$. Further from its form it is apparent that $V$ is a polynomial in $AHA^*H^{-1}$ and that $VH = HV = HV^*$, so that $VHV^* = H$. While we have used a special form of the pencil $AHA^* - xH$ it follows that similar results are true when the pencil is not in this form. Accordingly we have proved this theorem.

**Theorem 2.** Let $H$ be a nonsingular hermitian matrix and let $A$ be
a nonsingular matrix of the same order as $H$. Then there exists at least one matrix $V$, a polynomial in $AHA^*H^{-1}$, with the properties that $V^2 = E$, $VH$ is hermitian with the same signature as $H$ and that no characteristic number of $AHA^*(VH)^{-1}$ is negative. The matrix $A = DR$ where $DH$ is hermitian with the same signature as $H$ and $RHR^* = VH$. If all the characteristic numbers of $AHA^*H^{-1}$ are uniquely determined.

4. **Representation in the real field.** If $H$ is a real symmetric matrix and $A$ a real nonsingular matrix our argument may be carried through in the field of real numbers with only a few alterations. In (i), if the pair of conjugate elementary divisors is $(x - c \pm id)^n$, we take $H = qT$, $AHA^* = p(E + U)qT$, where

$$p = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Then, if

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix}^2 = p,$$

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix}GH$$

is symmetric and

$$\left\{ \begin{pmatrix} r & s \\ -s & r \end{pmatrix} \right\}^2 = AHA^*H^{-1}.$$ 

Case (ii) is unaltered. In (iii) we replace the matrices $D$ and $H$ of (9) by

$$\begin{pmatrix} 0 & bG \\ -bG & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix},$$

respectively. It therefore follows that Theorems 1 and 2 are true in the field of real numbers, if hermitian is replaced by symmetric and conjugate transposed by transposed.

5. **Conditions for the commutativity of the polar components.** Let the matrix $A$ have an $H$-representation $A = DR$. Then $A = RD_1$ where $D_1 = R^{-1}DR$. The matrix $D_1H$ is hermitian. Further $HA^*H^{-1}A = HD_1^*R^*H^{-1}RD_1 = D_1HH^{-1}D_1 = D_1^2$. Since $D_1^2 = AHA^*H^{-1}$, it follows that, if $A^*$ is commutative with $H^{-1}AH$, $D_1^2 = D^2$ and that for a proper choice $D_1 = D$. Conversely, if $D_1 = D$, $H^{-1}AH$ is commutative with $A^*$. We have therefore proved this theorem.
Theorem 3. If $A$ has an $H$-representation $A = DR$, $A$ also has an $H$-representation $A = RD$. A necessary and sufficient condition that $D = D_1$ or that $D$ be commutative with $R$ is that $A^*$ be commutative with $H^{-1}AH$.

Therefore, if $A = DR$ and $A$ is normal with respect to $H$, $A = RD$ so that $R$ and $D$ are commutative. That the converse is not true may be shown as follows. Let

$$H = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$AHA^*H^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The characteristic numbers of $AHA^*H^{-1}$ are all plus one and therefore $A$ has an $H$-representation. In fact

$$D = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

Hence

$$H^{-1}AH = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and is commutative with $A^*$. However, $H^{-1}AH$ is not a polynomial in $A^*$ and is therefore not normal with respect to $H$.

If the elementary divisors of $AHA^* - xH$ are all positive and linear and if $A = DR = RD$, it is comparatively easy to show that $H^{-1}AH$ is a polynomial in $A^*$ and therefore that $A$ is normal with respect to $H$.

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This seems to be the generalization of the classical result that a necessary and sufficient condition for the polar components of a matrix \( A \) to be commutative is that \( A \) be a normal matrix.

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REMARKS ON REGULARITY OF METHODS OF SUMMATION

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A doubly infinite matrix\(^1\) \((a_{mn}) (m, n = 1, 2, \cdots)\) is said to be regular, if for every sequence \( x = \{x_n\} \) with limit \( x' \) the corresponding sums \( y_m = \sum_{n=1}^{\infty} a_{mn} x_n \) exist for \( m = 1, 2, \cdots \), and if \( \lim_{m \to \infty} y_m = x' \).

An apparently more inclusive definition of regularity is that for each sequence \( x \) with limit \( x' \) the sums defining \( y_m \) shall exist for all \( m \geq m_0(x) \) and \( \lim_{m \to \infty} y_m = x' \). Tamarkin\(^2\) has shown that \((a_{mn})\) is regular in the latter sense if and only if there exists an \( m_1 \) independent of \( x \) such that the matrix \((a_{mn}) (m = m_1, n \geq 1)\) is regular in the former sense. Using point set theory in the Banach space \((c)\), he proves a theorem\(^3\) from which follows the result just mentioned. This note presents an elementary proof of that theorem and discusses some related topics.

**THEOREM 1.** Suppose the doubly infinite matrix \((a_{mn})\) has the property that for each sequence \( x = \{x_n\} \) with limit 0 there exists an \( m_0 = m_0(x) \) such that for all \( m \geq m_0(x) \), \( u_m = \lim \sup_{k \to \infty} \sum_{n=1}^{k} |a_{mn} x_n| < \infty \). Then there exists an \( m_1 \) such that \( \sum_{n=1}^{\infty} |a_{mn}| < \infty \) for all \( m \geq m_1 \).

If in addition \( \lim_{m \to \infty} u_m = 0 \) for each sequence \( x \) with limit 0, it will follow\(^4\) that there exists an \( N \) such that \( \sum_{n=1}^{\infty} |a_{mn}| \leq N < \infty \), for all \( m \geq m_1 \).

To prove Theorem 1, suppose there were an infinite sequence \( m_1 < m_2 < \cdots \) such that \( \sum_{n=1}^{\infty} |a_{mn}| = \infty \) for \( m \in \{m_r\} \). Let \( x_1, \cdots, x_{k_1} \) be chosen with unit moduli and with amplitudes such that

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\(^1\) In this note \( a_{mn}, x_n \) and \( x' \) denote finite complex numbers.


\(^3\) J. D. Tamarkin, loc. cit., p. 242, lines 1–6.

\(^4\) See, for example, I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, Journal für die reine und angewandte Mathematik, vol. 151 (1921), pp. 79–111; p. 85, Theorem 4.