ON MAJORANTS OF SUBHARMONIC AND ANALYTIC FUNCTIONS

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This paper represents a different approach to a whole group of problems connected with majorants of subharmonic functions. The same method has been used previously in order to prove a generalization of the Phragmén-Lindelöf theorem.\(^1\) It seems that the best approach is to prove first Lemma 4, and then the most important results are easily deducible. Corollary 6 is a generalization of a result of N. Levinson.\(^2\) His theorem has made me realize the importance of these results.

**Lemma 1.** If (i) \(0 < f(x) \leq 1\) and (ii) \(\int_a^b \log f(x) \cdot dx\) is finite, then

\[
\int_a^b \log \left(\int_{\xi}^x f(y)dy\right) dx
\]

is a continuous function of \(\xi\) in \((a, b)\).

We first suppose that \(f(x)\) is non-decreasing and that \((0, 1) = (a, b)\). We get

\[
\int_0^x f(y)dy > \int_{x/2}^x f(y)dy \geq (x/2)f(x/2).
\]

Hence

\[
\int_0^1 \log \left(\int_0^x f(y)dy\right) dx > \int_0^1 \log (x/2)dx + \int_0^1 \log f(x/2)dx
\]

\[
> 2 \int_0^1 \log x \cdot dx + 2 \int_0^1 \log f(x)dx
\]

\[
= -2 + 2 \int_0^1 \log f(x) \cdot dx.
\]

If \(f(x)\) is replaced by \(f(a + (b-a)x)\), we obtain

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\[
\int_a^b \log \left( \int_a^x f(y) \, dy \right) \, dx \geq -2(b - a) + (b - a) \log (b - a) \\
+ 2 \int_a^b \log f(x) \, dx.
\]

(3)

Next, if \( f(x) \) is general, we form the rearranged non-decreasing function \( \tilde{f}(x) \) for which

\[
\text{meas} \ E \left[ f(x) \leq y \right] = \text{meas} \ E \left[ \tilde{f}(x) \leq y \right]
\]

for all \( y \). We know that\(^3\)

\[
\int_0^1 \log f(x) \, dx = \int_0^1 \log \tilde{f}(x) \, dx,
\]

and that

\[
\left| \int_{x_0}^x f(y) \, dy \right| \leq \int_0^{x-x_0} \tilde{f}(y) \, dy.
\]

Hence, if \( x_0 \subset (0, 1) \),

\[
\int_0^1 \log \left| \int_{x_0}^x f(y) \, dy \right| \, dx \\
\geq \int_0^1 \log \left( \int_0^{x-x_0} \tilde{f}(y) \, dy \right) \, dx \\
= \int_0^{x_0} \log \left( \int_0^{x-x_0} \tilde{f}(y) \, dy \right) \, dx + \int_0^1 \log \left( \int_{x_0}^x \tilde{f}(y) \, dy \right) \, dx \\
= \int_0^{x_0} \log \left( \int_0^{x-x_0} \tilde{f}(y) \, dy \right) \, dx + \int_0^{1-x_0} \log \left( \int_0^x \tilde{f}(y) \, dy \right) \, dx \\
\geq 2 \int_0^1 \log \left( \int_0^x \tilde{f}(y) \, dy \right) \, dx.
\]

This, with (2) gives

\[
\int_0^1 \log \left( \int_{x_0}^x f(y) \, dy \right) \, dx \geq -4 + 4 \int_0^1 \log \tilde{f}(x) \, dx \\
= -4 + 4 \int_0^1 \log f(x) \, dx.
\]

In a similar fashion we get, for \( x_0 \subseteq (a, b) \),
\[
\int_a^b \log \left| \int_{x_0}^x f(y) \, dy \right| \, dx \geq -4(b - a) + 2(b - a) \log (b - a) + 4 \int_a^b \log f(x) \cdot dx.
\]
(4)

If \( \epsilon > 0 \) is given, we can find a \( \delta > 0 \) such that
\[
8\delta - 4\delta \log \delta - 4 \int_{x_0 - \delta}^{x_0 + \delta} \log f(x) \cdot dx < \epsilon / 3.
\]

Let us denote by \( J_1 \) the common part of \((a, b)\) and \((x_0 - \delta, x_0 + \delta)\), and by \( J_2 \) the rest of \((a, b)\). Then, by (3), for \( \xi \in J_1 \)
\[
0 > \int_{J_1} \log \left| \int_{\xi}^x f(y) \, dy \right| \cdot dx \geq -8\delta + 4\delta \log \delta + 4 \int_{x_0 - \delta}^{x_0 + \delta} \log f(x) \cdot dx > -\epsilon / 3.
\]

Further
\[
g_2(\xi) = \int_{J_2} \log \left| \int_{\xi}^x f(y) \, dy \right| \cdot dx
\]
is a continuous function for \( \xi = x_0 \) and, hence, there is a \( \delta_1 < \delta \) such that
\[
| g_2(\xi) - g_2(x_0) | \leq \epsilon / 3 \quad \text{for} \quad | \xi - x_0 | < \delta_1.
\]

Hence, if we call the integral (1) \( g(\xi) \), then
\[
| g(\xi) - g(x_0) | < \epsilon \quad \text{for} \quad | \xi - x_0 | < \delta_1.
\]

This shows that (1) is a continuous function of \( \xi \) at an arbitrary point \( x_0 \subseteq (a, b) \).

**Lemma 2.** Given a non-negative \( \psi(x) \subseteq L \), there is a domain \( D \), bounded by two continuous curves
\[
C_1 \equiv y = g_1(x), \quad C_2 \equiv y = g_2(x)
\]
and two straight lines \( x = x_0 - a, x = x_0 + a \), such that, if \( x + iy = f(re^{i\theta}) \) represents \( D \), conformally on the unit circle \( r < 1 \), then
\[
\frac{dx}{d\theta}_{r=1} \geq \text{const. exp } [\psi(x)]
\]
on \( C_1 \) and \( C_2 \).
Further the rectangle

(6) \[ R_0 : \left| y - y_0 \right| < a \log \frac{3}{2}, \quad \left| x - x_0 \right| < a \]

is interior to \( D \) and

\[ \left| y - y_0 \right| \leq c(a), \quad \left| x - x_0 \right| \leq a, \quad \text{with} \quad \lim_{a \to 0} c(a) = 0, \]

contains \( D \).

There is no loss of generality in supposing \( x_0 = y_0 = 0 \). We define \( D \) by constructing its conformal representation on the unit circle. First we define the boundary function of the harmonic function \( x(r, \theta) \) so that it satisfies the above condition for the derivative. We define it as the inverse function of

(7) \[ \theta(x) = b \int_{-a}^{x} e^{-\psi(t)} dt \]

and, for later convenience, we take \( b \) such that

(8) \[ b \int_{-a}^{a} e^{-\psi(t)} dt = \theta_0 < 2 \arccos \frac{3}{4} < \pi/3. \]

Then the inverse function \( x(1, \theta) \), defined in \((0, \theta_0) \) satisfies condition (5) and is continuous and decreasing. Further, we define \( x(1, \theta) = a \), for \( \theta \subset (\theta_0, \pi) \); \( x(1, \pi + \theta) = x(1, \theta_0 - \theta) \) for \( \theta \subset (0, \theta_0) \), and finally \( x(1, \theta) = -a \) for \( \theta \subset (\pi + \theta_0, 2\pi) \). In the interval \((\pi, \pi + \theta_0)\) condition (5) is again satisfied. Now, the boundary function is completely defined and

\[ x(r, \theta) = \frac{1 - r^2}{2\pi} \int_{0}^{2\pi} \frac{x(1, \phi) d\phi}{1 - 2r \cos (\theta - \phi) + r^2}. \]

The conjugate harmonic function is

\[ y(r, \theta) = -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{2r \sin (\theta - \phi)}{1 - 2r \cos (\theta - \phi) + r^2} x(1, \phi) d\phi. \]

By partial integration

\[ y(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| 1 - r e^{i(\theta - \phi)} \right| dx(1, \phi). \]

Since \( x \) is constant in some intervals, we get, using the above definition of \( x(1, \phi) \),
\[ y(r, \theta) = \frac{1}{2\pi} \int_0^{\theta_0} \log \left| 1 - re^{i(\theta - \phi)} \right| \]
\[ - \log \left| 1 + re^{i(\theta - \theta_0 + \phi)} \right| \, dx(1, \phi). \]

Now, the required conformal representation is given by \( x(r, \theta) + iy(r, \theta) \). The boundary curve of \( D \) in the parametric form is \( x(1, \theta) + iy(1, \theta) \), and we shall investigate it.

If \( \phi \leq 0, \theta_0 \), then \( x(1, \theta) \) runs from \(-a\) to \(+a\) and, since \( 1 - e^{i(\theta - \phi)} \)
\[ < 1 \quad \text{for} \quad \theta, \phi \in (0, \theta_0) \subset (0, \pi/3) \quad \text{(cf. (8))}, \]
\[ y(1, \theta) \leq -\frac{1}{2\pi} \int_0^{\theta_0} \log \left| 1 + e^{i(\theta - \theta_0 + \phi)} \right| \, dx \]
\[ \leq -\frac{1}{2\pi} \log \left| 1 + e^{i\theta_0} \right| \int_{-a}^{a} \, dx \]
\[ = -(a/\pi) \log 2 \cos \theta_0/2 < -(a/\pi) \log 3/2. \]

The first term \( I_1 \) on the right in (9) is more difficult to dispose of. We have

\[ I_1 = \frac{1}{2\pi} \int_0^{\theta_0} \log \left| 2 \sin \frac{\theta - \phi}{2} \right| \cdot dx(1, \phi) \]
\[ = \frac{1}{2\pi} \int_0^{\theta_0} \log \left| 2 \sin (\theta - \phi)/2 \right| \cdot dx(1, \phi) \]
\[ + \frac{1}{2\pi} \int_0^{\theta_0} \log \left| \theta - \phi \right| \cdot dx(1, \phi) = I'_1 + I'_1 , \]

Here, \( I'_1 \) is evidently a continuous function of \( \theta \) and

\[ 0 > I_1 > -\frac{a}{\pi} \log \frac{2 \sin \theta_0/2}{\theta_0} . \]

By means of (7), we change variables in \( I'_1' \) and obtain

\[ I_1'' = \frac{1}{2\pi} \int_{-a}^{a} \log \left| b \int_{\xi}^{x(\theta)} e^{-\psi(a)} \right| \, d\xi . \]

By Lemma 2, this is a continuous function of \( x \) and therefore also of \( \theta \) and from (3) we get

\[ 0 > I'_1'' > -\frac{2a}{\pi} + \frac{a \log 2a}{\pi} + \frac{1}{\pi} \int_{-a}^{a} \log \left| be^{-\psi(x)} \right| \, dx. \]
If we combine the last inequalities and introduce the abbreviation \( c(a) \), we get

\[- \frac{a}{\pi} \log 3/2 > y(1, \theta) > -c(a), \quad \theta \in (0, \theta_0),\]

with

\[
\lim_{a \to 0} c(a) = 0.
\]

From (9) it is easy to deduce that

\[ y(r, \theta + \pi) = -y(r, \theta_0 - \theta), \]

and we obtain

\[ \frac{a}{\pi} \log 3/2 < y(1, \theta) \leq c(a), \quad \theta \in (\pi, \pi + \theta_0). \]

Using the same notation we shall prove the following lemma.

**Lemma 3.** If \( \psi(x_0-a), \psi(x_0+a) \) are finite, then every subharmonic function \( \alpha(x, y) \), defined in \( D \), which satisfies

\[
\alpha(x, y) \leq e^{\psi(z)}, \quad |x| < a,
\]

has an upper bound in \( R \{ |y| < a \cdot \log 3/2, |x| < a \} \) which depends only on \( \psi(x) \).

If we represent \( D \) conformally on the unit circle \( C \), we get from \( \alpha(x, y) \) a subharmonic function \( \alpha(r, \theta) \) defined in \( C \). There, it must be less than any harmonic function with boundary values not less than those of \( \alpha(r, \theta) \). By (10), these are not greater than \( \exp \{ \psi(x(1, \theta)) \} \), where \( x(1, \theta) \) has been defined in the preceding lemma. Such a harmonic function is the Poisson integral of \( \exp \{ \psi(x(1, \theta)) \} \). We have to show that this Poisson integral is not identically equal to \( \infty \). A sufficient condition is

\[
\int_0^{2\pi} \exp \{ \psi(x(1, \theta)) \} \cdot d\theta < \infty.
\]

In view of the definition of \( x(1, \theta) \), this integral is obviously less than

\[
\left| \int_{-a}^{a} e^{\psi(x)} \frac{d\theta}{dx} \ dx \right| + \int_{\theta_0}^{\pi} e^{\psi(a)} d\theta + \left| \int_{-a}^{a} e^{\psi(x)} \frac{d\theta}{dx} \ dx \right| + \int_{\pi + \theta_0}^{2\pi} e^{\psi(-a)} d\theta,
\]

and using (5), also less than

\[
2b \int_{-a}^{a} e^{\psi(x)} e^{-\psi(z)} dx + \pi(e^{\psi(a)} + e^{\psi(-a)}) \leq 4ab + \pi(e^{\psi(a)} + e^{\psi(-a)}) < \infty.
\]
Hence \( \alpha(x, y) \) is, inside of \( D \), less than a finite harmonic function depending only on \( \psi(x) \). In the domain \( D \), completely interior to \( R \), it is therefore bounded from above by a constant depending only on \( \psi(x) \).

**Theorem.** If (i) \( \alpha(x, y) \) is subharmonic in \( R[|x|<c, |y|<d] \) and (ii) \( \alpha(x, y)<e^{\psi(x)}, \psi(x) \subset L, x \subset (-c, c), \psi(-c)<\infty, \psi(c)<\infty \), then for any \( \delta \), such that \( 0<\delta<d \), there is an upper bound \( C \) for \( \alpha(x, y) \) in \( D_0[|x|<c, |y|<d-\delta] \) dependent only on \( \delta \) and \( \psi(x) \), but independent of the particular \( \alpha(x, y) \).

In view of the Lemma 3, it is sufficient to show that there is an \( a \) and a finite number of domains \( D_k \) satisfying the conditions of this lemma, and, such that \( D_k \) contains (cf. (6)) \( R_k[|x-x_k|<a, |y-y_k| \leq c(a)] \), is contained in \( R \), and that \( \sum R_k \) covers completely \( D_0 \). Since \( \alpha(x, y) \) has a finite upper bound in every \( R_k \), it must be so also in \( D_0 \). And the upper bound will depend only on \( \psi(x) \) and \( D_0 \).

We determine \( a>0 \) such that \( c(a)-a \log 3/2<\delta/2 \). If we define \( D^*[|x|<c, |y|<d-\delta/2] \), then \( D_0 \subset D^* \subset R \), and, if \( R_k \subset D^* \), then the corresponding \( D_k \subset R \) (cf. (3.2)). Since \( R_k \) can be any rectangle, of the above size, in \( D^* \) and such that \( \psi(x_k+a) \) is finite, it is evident that we can find a finite number of them covering completely \( D_0 \). The theorem is proved.

**Corollary:** Let \( f(z) \) be a function, analytic in \( R[|x|<c, |y|<d] \), such that

\[
|f(z)| \leq M(x).
\]

If \( \int \log^+ \log^+ M(x)dx<\infty \), then for every domain \( D_0 \) completely interior to \( R \), there exists a \( \phi \) depending only on \( D_0 \) and \( M(x) \), such that

\[
|f(z)| \leq \phi \quad \text{for} \quad z \subset D_0.
\]

There is no loss of generality to suppose \( M(x)>e \). and then the sign + can be omitted over the log signs. If \( f(z) \) is analytic, then \( \log|f(z)| \) is subharmonic and the result follows from the preceding theorem.

Nils Sjöberg has proved the following theorem:

Let \( M(\theta) \) be given and \( 0<\epsilon<1 \). In order that the class of subharmonic functions, defined in \( |z|<1 \), which satisfy in \( 1-\epsilon \leq |z|<1 \)

\[
|\mu(re^{i\theta})| \leq M(\theta),
\]

\( ^4 \) Comptes Rendus du Congrès des Mathématiques à Helsinfors, 1938.
should be bounded from above in every circle $|z| \leq r_0 < 1$, it is sufficient that

$$\int_{-\pi}^{\pi} \log^+ M(\theta) \cdot d\theta < \infty.$$ 

This theorem can easily be deduced from our result. By $\xi = \log z$, we straighten out the concentric circles, and $1 - \epsilon \leq |z| < 1$ corresponds to $\log (1 - \epsilon) \leq R(\xi) < 0$. In this strip $\mu(\xi) \leq M(\theta)$, $\theta = \Im(\xi)$. Hence, if $M(\theta_0) < \infty$, then, by our result, the class of subharmonic functions $\mu(\xi)$ will be bounded in $\theta_0 \leq \arg z = \theta \leq \theta_0 + 2\pi$ and $\log (1 - \epsilon/2) \leq R(\xi) \leq \log (1 - \epsilon/4)$. Hence the same is true of $\mu(z)$ in $1 - \epsilon/2 \leq |z| \leq 1 - \epsilon/4$. But the class of subharmonic functions must have the same upper bound in $|z| \leq 1 - \epsilon/4$. The theorem is proved.

Similarly we can prove a generalization of the Phragmén-Lindelöf theorem:

If: (i) $f(z)$ is analytic in $\Re(z) > 0$, (ii) its boundary values on $\Re(z) = 0$ are in absolute value less than 1, (iii) there are two sequences $r_k \to \infty$ and $\epsilon_k \to 0$ such that

$$|f(z)| \leq \exp \{\epsilon_k r_k e^{\psi(\theta)}\}, \quad \psi(\theta) \subset L,$$

for $r_k(1 - \delta) < |z| < r_k$, then

$$|f(z)| \leq 1 \quad \text{for } \Re(z) > 0.$$

From condition (iii) it follows that

$$\log |f(r_k z)| / \epsilon_k r_k \leq e^{\psi(\theta)},$$

for $1 - \delta < |z| < 1$. By condition (ii), we can suppose $\psi(0) = \psi(\pi) = 0$. In the same way as in the preceding we deduce the existence of a $\phi$, such that

$$\log |f(r_k z)| / \epsilon_k r_k < \phi \quad \text{for } 1 - \delta/2 < |z| < 1 - \delta/4,$$

or

$$|f(z)| \leq \exp \{\epsilon_k r_k\}, \quad (1 - \delta/2) r_k < |z| < (1 - \delta/4) r_k.$$ 

Now we use the Phragmén-Lindelöf theorem in its classical form to deduce the desired result.

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