A NOTE ON APPROXIMATION BY RATIONAL FUNCTIONS

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The theory of the approximation by rational functions on point sets $E$ of the $z$-plane ($z = x + iy$) has been summarized by J. L. Walsh who himself has proved a great number of important theorems some of which are fundamental. The results concern both the case when $E$ is bounded and when $E$ extends to infinity.

In the present note a $L_p$-theory ($0 < p < \infty$) will be given for the following point sets extending to infinity:

A. The real axis $-\infty < x < \infty$, $y = 0$.
B. The half-plane $-\infty < x < \infty$, $0 < y < \infty$.

The only poles of the approximating functions are to lie at pre-assigned points whose number will be required to be as small as possible. We shall make use of the theory of the class $S_p$, the fundamental results of which are due to E. Hille and J. D. Tamarkin; $S_p$ is the set of functions $F(z)$ which, for $0 < y < \infty$, are regular and satisfy the inequality

$$\int_{-\infty}^{\infty} |F(x + iy)|^p dx \leq M^p \quad \text{or} \quad |F(z)| \leq M$$

for $0 < p < \infty$ or $p = \infty$, respectively, where $M$ depends on $F$ and $p$ only. By $|f(x+i y)|_p$ we denote

$$\left( \int_{-\infty}^{\infty} |f(x + iy)|^p dx \right)^{1/p} \quad \text{or} \quad \text{ess. u.b.} \ |f(x + iy)|$$

for $0 < p < \infty$ or $p = \infty$, respectively, and by $\alpha$ and $\beta$ two arbitrarily fixed points in the upper or lower half-plane, respectively. We obtain the following results:

**Theorem 1.** Let $0 < p < \infty$ and $F(t) \in L_p(-\infty, \infty)$, let $c$ be an integer greater than $p^{-1}$ and $r_k(z) = (\alpha - z)^k(\beta - z)^{-c-k}$ $[k = 0, \pm 1, \pm 2, \cdots]$.
Then there are finite linear combinations $s_n(z)$ of the $r_k(z)$ such that

$$|F(t) - s_n(t)|_p = \left( \int_{-\infty}^{\infty} |F(t) - s_n(t)|^{pt} dt \right)^{1/p} \to 0 \quad \text{as } n \to \infty.$$ 

Theorem 2. (a) Let $0 < p < \infty$ and $F(t) \in L_p(-\infty, \infty)$. A necessary and sufficient condition for the existence of rational functions $s_n(z)$ such that their only poles lie in a single point of the lower half-plane and that $|F(t) - s_n(t)|_p \to 0$ as $n \to \infty$ is that $F(t)$ is equivalent to the limit-function of an element $F(z)$ of $\mathcal{S}_p$.

(b) When the latter condition is satisfied then there are rational functions $s_n(z)$, with their only poles at $z = \beta$, such that, uniformly in the half-plane $0 < y < \infty$,

$$|F(x + iy) - s_n(x + iy)|_p \to 0 \quad \text{as } n \to \infty.$$ 

By a well known result\footnote{Since $s_n(t) \in L_p$, we have $s_n(z) \in \mathcal{S}_p$, $F(z) - s_n(z) \in \mathcal{S}_p$, and we can apply the Hille-Tamarkin Theorem 2.1 (iii), part 2, loc. cit.} concerning $\mathcal{S}_p$, 2(b) is a consequence of 2(a).

We start with giving explicit approximating functions in some special cases of problem (A), taking $\beta = \alpha$.

Theorem 1'. Let $F(t) \in L_1(-\infty, \infty)$ or $F(t) \in L_2(-\infty, \infty)$, or let $F(t)$ be continuous everywhere, including infinity.\footnote{A function $F(t)$ is said to be continuous at infinity when its limits, as $t \to \pm \infty$, both exist and are finite and equal.} Let $c = 2, 1, 0$ for $p = 1, 2, \infty$, respectively, and let

$$s_n(z) = \sum_{k=-n}^{n} a_k \frac{(\alpha - z)^k}{(z - \alpha)^{k+c}}, \quad a_k = \frac{i(\alpha - \bar{\alpha})}{2\pi} \int_{-\infty}^{\infty} F(t) \frac{(t - \alpha)^{k+c-1}}{(\alpha - t)^{k+1}} dt.$$ 

Then

$$|F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t)|_1 \quad \text{or} \quad |F(t) - s_N(t)|_2 \quad \text{or}$$

$$|F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t)|_\infty = \text{u.b.} \quad \frac{1}{N+1} \sum_{n=0}^{N} s_n(t),$$ 

respectively, tends to zero as $N \to \infty$. When $F(t)$ is continuous everywhere, including infinity, and of bounded variation in $(-\infty, \infty)$ then the $s_n(t)$ converge to $F(t)$ uniformly in $(-\infty, \infty)$.

It will suffice to take $\alpha = i$, the general case being deduced from this one by the substitution $t = \Im(\alpha) + t' \Im(\alpha)$. Let $F(t) \in L_2(-\infty, \infty)$, $t = \tan(1/2)\theta \quad [-\pi \leq \theta \leq \pi]$, and $f(\theta) = 2(1 + e^{2\theta})^{-1} F(\tan \theta/2)$. Then $F(t) \in L_2(-\infty, \infty)$ implies that $f(\theta) \in L_2(-\pi, \pi)$, and vice versa. Now
the Fourier series $\sum b_n e^{i n \theta}$, belonging to $f(\theta)$, converges to $f(\theta)$ in the mean square over $(-\pi, \pi)$. We have $e^{i \theta} = (i - t)(i + t)^{-1}$, $(1/2)(1 + e^{i \theta}) = i(i + t)^{-1}$; taking $a_n = ib_n$, we arrive finally at the required result. In a similar way we prove the remaining assertions of the theorem. We note that the sequence $\{(2i\pi)^{-1/2}(\alpha - \bar{\omega})^{1/2}(\alpha - \bar{\omega})^{-1}\}^{\infty}_{n=0}$ is a complete orthogonal and normal system with respect to $L_p(-\infty, \infty) [1 < p < \infty]$. 

To prove Theorem 1, we have to show that, given $\varepsilon > 0$, there is a finite linear combination $s_n(z)$ of the $r_n(z)$ such that $|F(t) - s_n(t)|_p < \varepsilon$. We can find a positive number $b$ and a function $f(t)$ such that $f(t)$ is zero for $|t| \geq b$ and continuous for $-b < t < b$, and that

$$\int_{-\infty}^{\infty} |F(t) - f(t)|_p dt \leq \delta, \quad \delta = \begin{cases} (\varepsilon/2)^p & \text{for } p > 1 \\ (1/2)\varepsilon^p & \text{for } p \leq 1. \end{cases}$$

The function $g(t) = (t - \beta)^p f(t)$ is continuous everywhere, including infinity. From results of Walsh we deduce the existence of functions $\sigma_n(z) = \sum_{k=-n}^{n} a_k,n (\alpha - \bar{k})^k$, $n = 0, 1, 2, \ldots$, $|g(t) - \sigma_n(t)|_\infty \to 0$ as $n \to \infty$. Taking $s_n(z) = (z - \beta)^{-n} \sigma_n(z)$, we have

$$|f(t) - s_n(t)|_p^p = \left|\frac{g(t)}{(t - \beta)^p} - \sigma_n(t)\right|_p^p \leq \left|g(t) - \sigma_n(t)\right|_\infty \int_{-\infty}^{\infty} \frac{dt}{|t - \beta|_p}. $$

The right side tends to zero as $n \to \infty$. Therefore, for some $n$, we have $|f(t) - s_n(t)|_p^p < \delta$, $|F(t) - s_n(t)|_p^p < \varepsilon^p$ which completes the proof.

To prove Theorem 2(a), we need some lemmas.

**Lemma 1.** Let $\varphi(w)$ belong to the Riesz class $H_p [0 < p < \infty]$, that is to say, let $\varphi(w)$ be regular for $|w| < 1$ and satisfy the inequality

$$\|\varphi(re^{i\theta})\|_p = \left(\int_{-\pi}^{\pi} |\varphi(re^{i\theta})|_p^p d\theta \right)^{1/p} \leq M, \quad 0 < r < 1,$$

where $M$ is independent of $r$.\(^{10}\) Then there are polynomials $P_n(w)$ $[n = 1, 2, \ldots]$ such that $\|\varphi(re^{i\theta}) - P_n(re^{i\theta})\|_p \to 0$ as $n \to \infty$, uniformly for $0 < r \leq 1$.

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\(^8\) Walsh, loc. cit. chap. 2, Theorem 16. It can also be deduced from Theorem 1' of this paper.

\(^9\) For $p = \infty$ the result holds if and only if $\varphi(e^{i\theta})$ is continuous for $-\pi \leq \theta \leq \pi$. Cf. Walsh, loc. cit., and Trans. Amer. Math. Soc. vol. 26 (1924) pp. 155-170.

\(^{10}\) F. Riesz, Math. Zeit. vol. 18 (1923) pp. 87-95.
By well known properties of the class \(H_p\), it will suffice to take \(r = 1\). Let \(\varphi(w) = \sum a_n w^n\). Since, for any fixed \(0 < R < 1\) and uniformly with respect to \(\theta\), \(-\pi \leq \theta \leq \pi\), the series \(\sum a_n R^n e^{i\theta}\) converges to \(\varphi(R e^{i\theta})\), the result can be deduced by means of the well known equation \(\|\varphi(e^{i\theta}) - \varphi(re^{i\theta})\|_p \to 0\) \([r \to 1]\).

**Lemma 2.** Let \(w = (i - z)(i + z)^{-1}\). The function \(F(z)\) belongs to \(\mathcal{S}_p\), if, and only if, the function \((1 + w)^{-2/p}\varphi(w)\) belongs to \(H_p\), where \(\varphi(w) = F(z)\).

Hille and Tamarkin have proved\(^{11}\) that the condition \(\varphi(w) \in H_p\) is necessary. To define the function \((1 + w)^{-2/p}\varphi(w)\), we cut the \(w\)-plane along the negative real axis from \(w = -1\) to \(w = -\infty\). When \(F(z)\) belongs to \(\mathcal{S}_p\) then its limit function \(F(t)\) \([y \to 0, x = t]\) belongs to \(L_p(-\infty, \infty)\), therefore \((1 + e^{i\theta})^{-2/p}\varphi(e^{i\theta})\) to \(L_p(-\pi, \pi)\). Let \(\varphi(w) = (1 + w)^{-2/p}\varphi(w)\), and \(0 < q < p/3\). By Hölder's theorem, we have

\[
\int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^{q/p} d\theta \leq \left( \int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^{q/p} \right)^{1-q/p} \left( \int_{-\pi}^{\pi} \frac{d\theta}{1 + r e^{i\theta}} \right)^{-q/p}.
\]

The right side is uniformly bounded for \(0 < r < 1\). Hence \(\varphi(w) \in H_q\); its limit-function \(\varphi(e^{i\theta})\), however, belongs to \(L_p(-\pi, \pi)\); hence\(^{12}\) \(\varphi(w) \in H_p\). Conversely, let \(\varphi(w) \in H_p\). From a result due to R. M. Gabriel\(^{13}\) we deduce that

\[
\int_C |\varphi(w)|^p \, dw \leq 2 \int_{-\pi}^{\pi} |\varphi(e^{i\theta})|^{q/p} d\theta,
\]

where \(C\) is any circle strictly interior to the unit circle \(\Gamma [|w| = 1]\). By Fatou's theorem, this inequality holds when \(C\) is a circle touching \(\Gamma\) from within at \(w = -1\). Finally, by the transformation \(w = (i - z)(i + z)^{-1}\), we deduce that \(|F(x + iy)|_p \leq 2^{q/p} ||\varphi(e^{i\theta})||_p\) \([0 < y < \infty]\) which proves the lemma. In a similar way we can show that when \(F(z) \in \mathcal{S}_p\) and \(F(t) \in L_q(-\infty, \infty)\) \([0 < p_q \leq \infty]\) then \(F(z) \in \mathcal{S}_q\).

**Lemma 3.** Let \(0 < p \leq \infty\), let \(f_n(z) \in \mathcal{S}_p\) \([n = 1, 2, \ldots]\), and let \(f_n(t)\) be the limit-function of \(f_n(z)\). Let \(F(t)\) be defined in \((-\infty, \infty)\) and \(|F(t) - f_n(t)|_p \to 0\) as \(n \to \infty\). Then \(F(t)\) is equivalent to the limit-function of an element \(f(z)\) of \(\mathcal{S}_p\).

\(^{11}\) Loc. cit. Lemma 2.5.


The proof for $0 < p < \infty$ is entirely different from that for $1 \leq p \leq \infty$, given in a former paper.\textsuperscript{14} Let $0 < p < 1$ and $\rho > 0$, and let $\phi(z) \in \mathcal{S}_p$. Then, for $\rho \leq y < \infty$, we have $|\phi(z)| \leq ((1/2)\pi\rho)^{-1/p} |\phi(t)|.^{15}$ Since $|f_n(t) - f_m(t)| \to 0 \text{ as } m, n \to \infty$, taking $\phi(z) = f_m(z) - f_n(z)$ we can deduce that the sequence $\{f_n(z)\}$ converges to an analytic function $f(z)$, uniformly for $-\infty < x < \infty$, $\rho < y < \infty$. Since there is a constant $K$, independent of $n$, such that $|f_n(t)| \leq K$, we have $|f_n(x+iy)| \leq K$ for any positive $y$. Hence $f(z) \in \mathcal{S}_p$. We are left to show that $f(t)$, the limit-function of $f(z)$, is equivalent to $F(t)$ in $(-\infty, \infty)$. Given $\epsilon > 0$, we have $|f_m(x) - f_N(x)|^p \leq \epsilon/12$ for $m \geq N$, fixing $N$ in a suitable way, and $|f_N(x+iy) - f_N(x)|^p \leq \epsilon/6$ for $0 < y \leq \delta = \delta(\epsilon, N)$. Hence

$$|f_m(x+iy) - f_m(x)|^p \leq |f_m(x+iy) - f_N(x+iy)|^p$$

$$+ |f_m(x) - f_N(x)|^p + |f_N(x+iy) - f_N(x)|^p \leq \epsilon/3$$

for $m \geq N$, $0 < y \leq \delta$, since the first term on the right side is not greater than the second term. Given $M > 0$, we have

$$\int_{-M}^{M} |f(x) - f_m(x)|^p dx \leq \int_{-M}^{M} |f(x+iy) - f_m(x+iy)|^p dx$$

$$+ |f(x+iy) - f(x)|^p + |f_m(x+iy) - f_m(x)|^p.$$ 

The right side is smaller than $\epsilon$ for $m \geq m_0(\epsilon)$, as we see fixing a suitable value for $y$. Consequently $f(x) = F(x)$ almost everywhere in any finite interval $(-M, M)$ and, therefore, in $(-\infty, \infty)$. With a slight alteration, the proof holds for $1 < p < \infty$.

By the lemma, the necessity of the condition in Theorem 2(a) is evident. For $s_n(t)$ belongs to $L_p(-\infty, \infty)$, therefore $s_n(z) \to s(z)$ in $\mathcal{S}_p$. To prove its sufficiency, we take first $1 < p < \infty$. By Theorem 1, there are rational functions $R_n(z)$ such that their only poles lie at $z = \beta$ and $z = \bar{\beta}$ and that $|F(t) - R_n(t)| \to 0$ as $n \to \infty$. Taking $R_n(z) = s_n(z) + \sigma_n(z)$, where the rational functions $s_n$ and $\sigma_n$ vanish at infinity and have no poles other than at $z = \beta$ or $z = \bar{\beta}$, respectively, we have $s_n(z) \in \mathcal{S}_p$, $\sigma_n(z) \in \mathcal{S}_p$. Denoting by $\mathcal{S}$ the Hilbert operator

$$\mathcal{S}(f) = \frac{1}{\pi PV} \int_{-\infty}^{\infty} \frac{f(t) dt}{t - x},$$

we have $|\mathcal{S}f| \leq C_p |f|$, $\mathcal{S}F = iF(x)$ and $\mathcal{S}s_n = is_n(x)$, $\mathcal{S}\sigma_n = -i\sigma_n(x).^{14}$


\textsuperscript{15} This can be shown by means of the inequality (73), M. Plancherel and G. Polya, Comment. Math. Helv. vol. 10 (1937-1938) pp. 110-163.
Hence
\[
2 | F(t) - s_n(t) |_p = | iF + \delta F - (iR_n + \delta R_n) |_p \\
\leq | F - R_n |_p + | \delta (F - R_n) |_p \leq (C_p + 1) | F - R_n |_p
\]
which tends to zero as \( n \to \infty \). Hence \( | F(t) - s_n(t) |_p \to 0 \) as \( n \to \infty \).

Let now \( 0 < p \leq 1 \) and \( F(z) \in \mathbb{S}_p \), let \( \beta = -i, z = i(1-w)(1+w)^{-1} \) and \( \varphi(w) = F(z) \). Given \( \epsilon > 0 \), from the Lemmas 2 and 1 we infer the existence of a polynomial \( P(z) \) such that
\[
\int^{\pi} -\pi | \varphi(t) (1 + e^{i\theta})^{-2/p} - P(e^{i\theta}) |^p d\theta \leq \epsilon / 4.
\]
Hence
\[
\int^{\infty}_{-\infty} | F(t) - (1 + e^{i\theta})^{2/p} P\left( \frac{i - t}{i + t} \right) |^p dt \leq \epsilon / 2,
\]
where \( t = \tan \theta / 2 \). Let \( b \) be an integer, \( p^{-1} < b \leq 1 + p^{-1} \). Then the rational function \( \chi(z) = (2i)^{b} (i + z)^{-b} P \{ (i - z)(i + z)^{-1} \} \) has no singularity except at \( z = -i \). Since \( \chi(t) \in L_p(-\infty, \infty) \), we have \( | \chi(t) |_p = C < \infty \). Now the function \( (1 + e^{i\theta})^{2/p-1} \) can be approximated by polynomials \( Q_n(e^{i\theta}) \) \( [m = 1, 2, \cdots] \), uniformly for \( -\pi \leq \theta \leq \pi \). Hence, for some \( m \), we have
\[
\int^{\infty}_{-\infty} (1 + e^{i\theta})^{2/p} P\left( \frac{i - t}{i + t} \right) - \chi(t) Q_m\left( \frac{i - t}{i + t} \right) |^p dt < \epsilon / 2.
\]
Thus \( | F(t) - \chi(t) Q_m \{ (i - t)(i + t)^{-1} \} |_p < \epsilon^{1/p} \). This completes the proof which, slightly altered, holds for \( 1 < p \leq 2 \).

For \( p = 1, 2, \infty \), we obtain explicit approximating functions by Theorem 1' and by the lemma:

Let \( 1 \leq p \leq \infty \) and \( F(z) \in \mathbb{S}_p \), let \( a \) be an integer and \( a \geq 0 \) for \( p = 1 \), \( a \geq 2 \) for \( p = \infty \), \( a \geq 1 \) otherwise; then
\[
\int^{\infty}_{-\infty} F(t) \frac{(\alpha - t)^n}{(t - \beta)^{n+a}} dt = 0 \quad \text{for } n = 0, 1, 2, \cdots.
\]

**Theorem 2'**. Let \( p = 2, 1, \) or \( \infty \) and \( c = 1, 2, \) or \( 0 \), respectively; let \( F(z) \in \mathbb{S}_p \) and \( F(t) \), the limit-function of \( F(z) \), be continuous everywhere including infinity when \( p = \infty \). Let \( s_n(z) \) be defined by
\[
\sum_{k=0}^{n} a_k \frac{(\beta - z)^k}{(z - \beta)^{k+c}} \left[ \begin{array}{c} p = 2 \end{array} \right] \quad \text{or} \quad \frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=0}^{j} a_k \frac{(\beta - z)^k}{(z - \beta)^{k+c}} \left[ \begin{array}{c} p = \infty \end{array} \right],
\]
where
\[ a_k = i \frac{\beta - \beta}{2\pi} \int_{-\infty}^{\infty} F(t) \frac{(t - \beta)^{k+1}}{(\beta - t)^{k+1}} \, dt. \]

Then, uniformly for \( 0 \leq y < \infty \), \( |F(x + iy) - s_n(x + iy)| \to 0 \) as \( n \to \infty \).

Applying Theorem 2' to the components of \( g(z) = (1/2)^{(1-s)} \Gamma((1/2)s)\pi^{-s/2} \xi(s) \),\(^1\) where \( \xi(s) \) is the Riemann zeta-function and \( z = i(l - 2s) \), we can deduce the following corollary:

Let \( 0 \leq a < \infty \), \( q = i(1-a) \), \( r = i(1+a) \), let
\[ \vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}; \quad b_0 = \vartheta(1)/2 + (1 - a/2) \int_{1}^{\infty} v^{a/4} \vartheta'(v) \, dv; \]
\[ L_n^{(f)}(x) = \sum_{k=0}^{n} C_{n+j,k+j} \frac{(-x)^k}{k!}; \]
\[ b_n = (-1)^n \int_{1}^{\infty} v^{a/4} \vartheta'(v) \left\{ L_n^{(0)}(\log v)/2 - (a/2)L_n^{(-1)}(\log v)/2 \right\} \, dv, \]
\[ n > 0. \]

Then the series
\[ \sum_{n=0}^{\infty} b_n \left\{ \left( \frac{q - z}{r + z} \right)^n + \left( \frac{q + z}{r - z} \right)^n \right\} \]
converges to \( g(z) \) uniformly for \( -\infty < x < \infty \), \( -a \leq y \leq a \), while it does not converge whenever \( |y| > a \).

The series takes a simple form for \( a = 0 \) (critical line).

\(^1\) In fact to the function \( g(z) = g(\gamma \mp ia) \in \mathbb{S}_\alpha \), where \( g(z) = g_1(z) + g_2(\mp z) \), \( g_1(z) = ((1 + z^2)/16) \int_{1}^{\infty} \vartheta(t-1) \vartheta(it) (1/4) \, dt - (1/4) \vartheta(1) \).