ON FIBRE SPACES

is of course the multiplication in $A_1$). By the proof of Theorem 2, in order to show the equivalence of $A$ and $A_1$ it is sufficient to show that $[w, w] = \gamma f$ and $[w, z] = [zU, w]$ for every $z$ of $R$. But $[w, w] = w(f^{-1}w) = (fg)(f^{-1}fg) = fg^2 = \gamma f$, and $[w, z] = w(f^{-1}z) = (fg)(f^{-1}fx) = (fg)x = g(x \cdot fS) = (f \cdot xS)g = (f \cdot xS)(f^{-1}fg) = zU(f^{-1}w) = [zU, w]$. This proves the theorem.

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ON FIBRE SPACES. I

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In subsequent papers I propose to investigate various properties of fibre spaces. The object of the fundamental Hurewicz-Steenrod definition is to state a minimum set of readily verifiable conditions under which the covering homotopy theorem holds. An apparent defect of their definition is that it is not topologically invariant. In fact, for topological space $X$ and metrizable non-compact space $B$ the property "$X$ is a fibre space over $B"$ depends on the metric of $B$. The object of this note is to give a topologically invariant definition of fibre space and to show that (when $B$ is metrizable) $X$ is a fibre space over $B$ in this sense if and only if $B$ has a metric in which $X$ is a fibre space over $B$ in the sense of Hurewicz-Steenrod. Since the definition of fibre space is controlled by the covering homotopy theorem, an essential part of my program is to give a topologically invariant definition of uniform homotopy.

Let $\pi$ be a continuous mapping of a topological space $X$ into another topological space $B$. Let $\Delta = \Delta(B)$ denote the diagonal set $\sum_{b \in B} (b, b)$ of the product space $B \times B$ and let $\bar{\pi}$ denote the mapping of $X \times B$ into $B \times B$ which is induced by the mapping $\pi$ according to the rule $\bar{\pi}(x, b) = (\pi(x), b)$. Thus the graph $G$ of $\pi$ is the set $\bar{\pi}^{-1}(\Delta)$, and $\bar{\pi}^{-1}(U)$ is a neighborhood of $G$ whenever $U$ is a neighborhood of $\Delta$.

Any neighborhood $U$ of $\Delta$ determines uniquely a covering of $B$ by neighborhoods $N_U(b)$ according to the rule $b' \in N_U(b)$ when $(b, b') \in U$.

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2 How well they succeeded in this will be indicated in my next communication.
However not every covering of $B$ by neighborhoods need arise in this fashion—although the star neighborhoods of any open covering of $B$ may always be so generated.

A slicing function $\phi$ for $\pi$ is any continuous mapping defined over $\tilde{\pi}^{-1}(U)$ for some neighborhood $U$ of $\Delta$, with values in $X$, which satisfies the conditions

$$ \pi \phi(x, b) = b, $$
$$ \phi(x, \pi(x)) = x, $$

whenever $\phi$ is defined. I shall call $\pi$ a fibre mapping relative to $U$ if it has a slicing function defined over $\tilde{\pi}^{-1}(U)$. If $\pi$ is a fibre mapping I shall say that $X$ is a fibre space over the subset $\pi(X)$ of $B$. Since $U$ is a neighborhood of $\Delta$, $\pi(X)$ is open and closed in $B$.

This new definition is equivalent to the old one if the base space is compact metric (so that the Hurewicz-Steenrod definition is topologically invariant in this case). In fact, for metric space $B$, let $\sigma_e$ denote that neighborhood of $\Delta$ which determines the covering of $B$ by $e$-spheres. Clearly $X$ is a fibre space (relative to $\pi$) over the metric space $\pi(X)$ in the sense of Hurewicz-Steenrod if and only if $\pi$ has a slicing function defined over $\tilde{\pi}^{-1}(\sigma_e)$ for some $e > 0$. Hence, if $\pi$ is a fibre mapping and $\pi(X)$ is compact metrizable then $X$ is a fibre space over $\pi(X)$ in the sense of Hurewicz-Steenrod no matter how $\pi(X)$ is metrized.

Now let $B$ denote an arbitrary metrizable space, let $U$ be an open neighborhood of $A(B)$ and let $\pi$ be a fibre mapping whose slicing function is defined over $\tilde{\pi}^{-1}(U)$. For simplicity, assume also that $\pi(X) = B$. To show that $X$ is a fibre space in the sense of Hurewicz-Steenrod when $B$ is properly metrized it is clearly sufficient to so metrize $B$ that $\sigma_e \subset U$ for some $e > 0$.

**Lemma.** If $B$ is metrizable and $U$ is an open neighborhood of $\Delta(B)$ then $B$ can be so metrized that $\sigma_1 \subset U$.

Choose any random metric $d$ for $B$. Since $B \times B$ is metric, hence normal, it is possible to define a continuous function $f \in [0, 1]^{B \times B}$ such that

$$ f(b, b_0) = \begin{cases} 0 & \text{when } (b, b_0) \in \Delta, \\ 1 & \text{when } (b, b_0) \in B \times B - U. \end{cases} $$

Let $\phi$ denote the (continuous) mapping $b \rightarrow f_b$, where $f_b(b_0) = f(b, b_0)$.

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8 This proof is modelled after a proof in André Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Scientifiques et Industrielles, no. 551, 1938, p. 15.
The graph $B' = \sum \sigma(b, \phi(b))$ of $\phi$ is homeomorphic to $B$. The metric of $B'$ is induced by the metric of the product $B \times [0, 1]^B$ and is given by the formula

$$\delta(b_1', b_2) = \left\{d^2(b_1, b_2) + d^2(\phi(b_1), \phi(b_2))\right\}^{1/2},$$

where $b$ and $b'$ denote corresponding points of $B$ and $B'$. If now $(b_1', b_2') \in \sigma_1$ then $\delta(b_1', b_2') < 1$, hence $d(\phi(b_1), \phi(b_2)) < 1$, hence $\sup_{b' \in B} |f(b_1, b) - f(b_2, b)| < 1$. It follows that $f(b_1, b_2) = |f(b_1, b_2) - f(b_2, b_2)| < 1$, so that $(b_1, b_2) \in U$ and $(b_1', b_2') \in U'$.

**Theorem.** If $\pi$ is a fibre mapping and $B$ is metrizable then the metric of $B$ can be so chosen that $X$ is a fibre space over $\pi(X)$ (relative to $\pi$) in the sense of Hurewicz and Steenrod.

I conclude by defining uniform homotopy and stating the covering homotopy theorem for general fibre spaces. If $h$ is a homotopy in $B$ of a space $Y$ and $U$ is a neighborhood of $\Delta$ I shall say that $h$ is uniform with respect to $U$ if there is a $\delta > 0$ such that $|t - t'| < \delta$ implies that $\sum_{y \in Y} h(y, t', y(t)) \subset U$. Let $E_\delta = \sum_{0 \leq t, t' \leq 1, |t - t'| < \delta} \sum_{y \in Y} \sigma(h(y, t), h(y, t'))$, so that $E_\delta \subset \Delta$ and $E_1 \subset B \times B$. Clearly the neighborhoods $U$ with respect to which $h$ is uniform are those which contain an $E_\delta$ for some $\delta > 0$. Thus $h$ is always uniform with respect to $B \times B$; in the event that $Y$ is compact $h$ is uniform with respect to every neighborhood $U$. I shall call a homotopy $h^*$ in $X$ a covering homotopy (with respect to $\pi$) if

1. $\pi h^* = h$,
2. $h_{[0, 1]}(y)$ degenerates to a point whenever $h_{[0, 1]}(y)$ degenerates to a point.

I shall refer to the mappings $h_0$ and $h_{[0, 1]}^*(y)$ as the initial values of the homotopies $h$ and $h^*$, respectively. With these notations the covering homotopy theorem for fibre mappings reads thus.

**Theorem.** Given a fibre mapping $\pi \in B^X$ relative to $U$, a mapping $g \in X^Y$ and a homotopy $h$ in $B$, uniform with respect to $U$, with initial value $\pi g$, there exists a covering homotopy $h^*$ in $X$ with initial value $g$.

The covering homotopy $h^*$ is constructed stepwise and is easily seen to be uniform with respect to $U^* = \pi^{-1}(U)$ where $\pi(x, x') = (\pi(x), \pi(x'))$. Of course if $U$ is a $\sigma$, the neighborhood $U^*$ of $\Delta(X)$ need not be a $\sigma_*(X)$.

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