A CONJECTURE OF ORE ON CHAINS IN
PARTIALLY ORDERED SETS

SAUNDERS MACLANE

In a recent investigation, Ore has given a form of the Jordan-Hölder theorem valid for an arbitrary partially ordered set $P$. This theorem involves essentially the deformation of one chain into another by successive steps, each step being like that used in the conventional Jordan-Hölder theorem. Ore observes that his first theorem would be slightly easier to apply if it were proved under a weaker hypothesis. The modified theorem runs as follows:

**THEOREM.** If $P$ is a partially ordered set in which every chain joining two elements is finite, then any complete chain between two elements $b < a$ can be deformed into any other complete chain between the same two elements.

The proof rests on this lemma:

**LEMMA.** Under the hypothesis of the theorem, if $C$ is a complete chain from $b$ to $a$ which cannot be deformed into the complete chain $D$ from $b$ to $a$, there exist in $P$ elements $b' < a'$ and complete chains $C'$ and $D'$ from $b'$ to $a'$ such that $C'$ cannot be deformed into $D'$ and such that $b \leq b'$, $a' \leq a$ where either $b < b'$ or $a' < a$.

**PROOF.** Case 1. $C$ and $D$ have in common the element $e$, $b < e < a$. Then either $C$ cannot be deformed into $D$, or $C$ cannot be deformed into $D$. In these two cases, set $b' = b$, $a' = e$ or $b' = e$, $a' = a$, respectively.

Case 2. $C$ and $D$ have no elements in common. Since $C$ cannot be deformed into $D$, they cannot together constitute a simple cycle. There will then exist, say, elements $c$ in $C$ and $d$ in $D$ with $b < c < a$, $b < d < a$ and an element $m$ in $P$ with $c \leq m < a$, $d \leq m < a$. Because of the hypothesis that every chain in $P$ joining two elements is finite, there will exist in $P$ finite complete chains $E_m$, $F_m$, $G_m$. Then $b$ is joined to $a$ by four complete chains,

\[
C_b^e + C_a^c, \quad C_b^e + F_m^m + E_m^c, \\
D_b^d + G_d^m + E_m^a, \quad D_b^d + D_d^a.
\]

Received by the editors December 15, 1942.


2 Terminology and notation follow the paper of Ore.
Since $C$ cannot be deformed into $D$, one of the following three deformations must be impossible:

$$
C_e \rightarrow F_e^m + E_m^a, \quad C_b^c + F_b^m \rightarrow D_b^d + G_d^m,
$$

$$
G_d^m + E_m^a \rightarrow D_d^a.
$$

In the first case we set $a' = a$, $b' = c$; in the second case, $a' = m$, $b' = b$; in the third case $a' = a$, $b' = d$. In each case we have the conclusion of the lemma.

To prove the theorem, suppose that $P$ were to contain two complete chains $C$ and $D$ joining $b$ to $a$ in such wise that $C$ cannot be deformed into $D$. By induction on $n$, the lemma gives in $P$ elements $a = a_0 \geq a_1 \geq \cdots \geq a_n$ and $b = b_0 \leq b_1 \leq b_n \leq a_n$ such that for each $i$ either $a_{i-1} > a_i$ or $b_{i-1} < b_i$ $(i = 1, \ldots, n)$, and such that there are complete chains $C_n$, $D_n$ joining $b_n$ to $a_n$ with $C_n$ not deformable into $D_n$. This construction can be carried on indefinitely, using the axiom of choice to select at each stage a definite pair $a_{n+1}$, $b_{n+1}$. This produces two sequences of elements $a_i$, $b_i$ with

$$
b_0 \leq b_1 \leq b_2 \leq \cdots \leq a_2 \leq a_1 \leq a_0.
$$

Furthermore, the inequality sign holds an infinite number of times here, so that we obtain an infinite chain joining $b = b_0$ to $a = a_0$, contrary to the hypothesis of the theorem.

Harvard University