NEW SYSTEMS OF HYPERGEODESICS DEFINED
ON A SURFACE

P. O. BELL

Introduction. Let a non-ruled surface $S$ be referred to its asymptotic net as parametric. As a point $P_y$ moves along a curve $C_\lambda$ of $S$, the tangents at $P_y$ to the $u$- and $v$-asymptotic curves of $S$ describe two ruled surfaces $R_u^\alpha$ and $R_v^\beta$, respectively. Let $S_p$ and $S_\sigma$ denote arbitrary transversal surfaces of the congruences of $u$- and $v$-tangents of $S$, respectively. The purpose of the present paper is to introduce and study systems of curves of $S$ which will be called $\rho$- and $\sigma$-tangeodesics.

Definition. A curve $C_\lambda$ of $S$ whose associated ruled surface $R_u^\alpha$ intersects the surface $S_p$ in an asymptotic curve of $R_u^\alpha$ is a $\rho$-tangeodesic of $S$. Similarly, a curve $C_\lambda$ of $S$ whose associated ruled surface $R_v^\beta$ intersects $S_\sigma$ in an asymptotic curve of $R_v^\beta$ is a $\sigma$-tangeodesic of $S$.

The $\rho$- and $\sigma$-tangeodesics of $S$ at $P_y$ are found to be associated in remarkable manners with the edges of Green, the directrices of Wilczynski, and the projective normal of Fubini. In fact, a new geometric characterization is obtained for each of these lines.

1. Tangeodesics. If the parametric net on a non-ruled surface $S$ is the asymptotic net, the homogeneous projective coordinates $y^i(u, v)$ $(i = 1, 2, 3, 4)$ of a general point $P_y$ of $S$ are solutions of a system of differential equations which may be assumed to be reduced to Wilczynski's canonical form

\[ y_{uu} + 2by_v + fy = 0, \quad y_{vv} + 2a'y_u + gy = 0. \]

The homogeneous coordinates of points $\rho, \sigma$ on arbitrarily selected transversal surfaces $S_p$ and $S_\sigma$ of the congruences of $u$- and $v$-tangents of $S$ are given by the vector forms

\[ \rho = y_u - \beta y, \quad \sigma = y_v - \alpha y, \]

wherein $\beta, \alpha$ are arbitrary analytic functions of $u, v$.

Let $l$ denote the line joining $\rho, \sigma$ and let $l'$ denote its reciprocal at $P_y$. The line $l'$ joins the points $P_y$ and $z$ where $z$ is given by

\[ z = y_{uv} - \alpha y_u - \beta y_v, \]

in which $\beta$ and $\alpha$ are the functions in (1.2). The line $l$, according to Green's classification, is a line of the first kind and generates a con-
gruence $\Gamma$ of the first kind as $P_y$ moves over $S$. The line $l'$ is a line of the second kind and generates a congruence $\Gamma'$ of the second kind as $P_y$ moves over $S$.

Let $C_\lambda$ denote an integral curve of the curvilinear differential equation

$$ (1.4) \quad dv - \lambda(u, v)du = 0. $$

Regarding $u$ as independent variable we write $v' = \lambda(u, v)$ and $v'' = \lambda_u + \lambda_v$, in which accents indicate differentiation with respect to $u$.

The homogeneous coordinates of a general point of the ruled surface $R_\lambda^u$ are represented by the vector form

$$ (1.5) \quad \tilde{y} = y_u + wy, $$

wherein $u$ and $w$ are independent variables and $v$ varies in accordance with the relation $v' = \lambda(u, v)$.

Let us put $u = u(t), w = w(t)$, so that $\tilde{y}$ describes a curve on $R_\lambda^u$ as $t$ varies. The necessary and sufficient condition that this curve be an asymptotic curve of $R_\lambda^u$ is that the determinant equation

$$ (1.6) \quad \begin{vmatrix} \tilde{y} & y_u & v'y & v'w \\ \ddot{y} & y_{uu} & v''y & v'w \\ \ddot{y} & y_{uw} & v'w & v''y \\ v'' & v' & v & b \\ \end{vmatrix} = 0 $$

be satisfied. If we transform equation (1.6) by making use of equations (1.5), (1.4) and (1.1) we obtain, in view of the inequality $(\tilde{y}, y_u, y_v, y_{uw}) 
eq 0$, the equation

$$ (1.7) \quad \frac{dw}{du} = \begin{vmatrix} 2b^2 + (b_u - 2bw)v' + (w^2 + 2b_v + f)v'' - 2a'bv^3 & -bvo'' \\ 2a'v'v'' & -2b^2 \end{vmatrix} / v''^2. $$

As $P_y$ moves along $C_\lambda$ the point $p$ moves in the direction defined by (1.7) if and only if $w = -\beta$ satisfies (1.7). To obtain, therefore, the curvilinear differential equation for the $\rho$-tangeodesics we have merely to substitute $-\beta$ for $w$ in (1.7) and clear of fractions. The result is

$$ (1.8) \quad bv'' - 2b^2 - (2b\beta + b_u)v' - (\beta^2 + 2b_v + f + \beta_u)v'' + (2a'b - \beta_v)v'^2 = 0. $$

The differential equation for the $\sigma$-tangeodesics may be obtained by making the substitution

$$ \begin{pmatrix} v'' & v' & v & b & a' & \beta & f \\ -v''/v'^3 & 1/v' & u & a' & b & \alpha & g \end{pmatrix} $$

in (1.8). The result, on simplifying, is

$$ (1.9) \quad a'v'' + \alpha_u - 2a'b + (\alpha^2 + 2a'a + g + \alpha) v' + (2a'\alpha + a')v'^2 + 2a''v'^3 = 0. $$

2. Systems of hypergeodesics which have no cusp-axes. The curves defined on a surface $S$ by a differential equation of the form
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(2.1) \[ v'' = A + Bv' + Cv'^2 + Dv'^3 \]

in which the coefficients are functions of \( u, v \) and accents indicate differentiation with respect to the independent variable \( u \), are called hypergeodesics. The envelope of the osculating planes at a point \( P_y \) of the hypergeodesics (2.1) is a cone which is ordinarily of the third class. When this cone is of the third class it has three distinct cusp-planes which intersect in a line called the cusp-axis of the cone, or the cusp-axis of the hypergeodesics at the point \( P_y \). The cusp-axis is the line \( l' \) for which \( \alpha \) and \( \beta \) are given by

(2.2) \[ \alpha = C/2, \quad \beta = -B/2. \]

We are interested here in those cases in which the class of the cone is less than three and the cone has no cusp-axis. The local equation of the osculating plane at \( P_y \) of the curve \( \zeta \) defined by (1.4) is

(2.3) \[ 2\lambda(\lambda x_2 - x_3) + (\lambda' - 2b + 2a'\lambda^2)x_4 = 0, \]

when referred to the tetrahedron whose vertices have the general coordinates \( y, y_u, y_v, y_{uv} \). Assuming \( \zeta_\lambda \) to be an integral curve of (2.1) we replace \( \lambda' \) by the right member of (2.1) and put \( \lambda \) in place of \( v' \). The result is

(2.4) \[ 2\lambda(\lambda x_2 - x_3) + (A - 2b + B\lambda + C\lambda^2 + [D + 2a']\lambda^3)x_4 = 0. \]

The union curves of a congruence \( \Gamma' \) form a system of hypergeodesics, sometimes called an axial system, whose osculating planes at \( P_y \) all pass through the line \( l' \) of the congruence \( \Gamma' \). Equation (2.1) represents such a system if \( A = 2b \) and \( D = -2a' \).

A system of hypergeodesics (2.1) for which

(2.5) \[ A = 2b, \quad D \neq -2a' \]

will be called, for reasons which appear later, a \( u \)-polar system; and a system (2.1) for which

(2.6) \[ D = -2a', \quad A \neq 2b \]

will be called a \( v \)-polar system.

If system (2.1) is a \( u \)-polar system, the equation for the envelope of its osculating planes at \( P_y \) may be readily found from (2.4) to be

(2.7) \[ (2x_2 + Cx_4)^2 - 4(D + 2a')(Bx_4 - 2x_3)x_4 = 0. \]

Similarly, if (2.1) represents a \( v \)-polar system the equation for the envelope of its osculating planes at \( P_y \) may be found to be

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\[(2.8) \quad (2x_3 - Bx_4)^2 - 4(A - 2b)(2x_3 + Cx_4)x_4 = 0.\]

Since the cones (2.7) and (2.8) are nondegenerate quadric cones, they have no cusp-axes at \(P_y\). Hence we have that neither a \(u\)-polar system nor a \(v\)-polar system of hypergeodesics has a cusp-axis at \(P_y\).

There are two generators of the cone (2.7) which are such that the tangent planes of the cone along these generators pass through the \(u\)-tangent to \(S\) at \(P_y\). One of these is the \(v\)-tangent to \(S\) at \(P_y\) and the other is the line \(l'\) for which

\[(2.9) \quad \alpha = C/2, \quad \beta = -B/2,\]

wherein \(B\) and \(C\) are the functions appearing in (2.7). This line \(l'\) will be called the \(u\)-edge of the \(u\)-polar system.

The \(v\)-edge of a \(v\)-polar system is characterized similarly.

Since equations (2.9) are of the same form as equations (2.2), we have immediately this theorem.

**Theorem 2.1.** If the coefficients \(B\) and \(C\) of the differential equation of a non-polar system of hypergeodesics are identical with the corresponding coefficients of the differential equation of a \(u\)-polar system of hypergeodesics, the cusp-axis of the non-polar system at \(P_y\) coincides with the \(u\)-edge of the \(u\)-polar system at \(P_y\).

A similar theorem applies, of course, to a \(v\)-polar system of hypergeodesics.

The forms of the differential equations (1.8) and (1.9) show clearly that the \(\rho\)- and \(\sigma\)-tangeodesics form \(u\)- and \(v\)-polar systems of hypergeodesics. For the system (1.8) we have

\[(2.10) \quad A = 2b, \quad B = 2\beta + b_u/b, \quad C = (\beta^2 + 2\beta_v + f + \beta_u)/b, \quad D = (\beta_v - 2a'b)/b.\]

For the system (1.9) we have

\[(2.11) \quad A = (2a'b - \alpha_u)/a', \quad B = - (\alpha^2 + 2a'_u + g + \alpha_v)/a', \quad C = - 2\alpha - a'_v/a', \quad D = - 2a'.\]

The cone (2.7) is associated with the system (1.8) of \(\rho\)-tangeodesics if \(A, B, C, D\) are given by (2.10). Similarly, if \(A, B, C, D\) are defined by (2.11), the cone (2.8) is associated with the system (1.9) of \(\sigma\)-tangeodesics.

The \(u\)-edge of the \(\rho\)-tangeodesics (1.8) is the line \(l'_1\) passing through the points \(P_y\) and \(z_1\), where \(z_1\) is given by \(z_1 = y_u - \alpha_1 y_u - \beta_1 y_v\), in which

\[(2.12) \quad \beta_1 = - \beta - b_u/2b, \quad \alpha_1 = (\beta^2 + 2\beta_v + f + \beta_u)/2b.\]
The \(v\)-edge of the \(\sigma\)-tangeodesics is the line \(l_z\) passing through \(P_y\) and \(z_2\) where \(z_2\) is given by \(z_2 = y_{uv} - \alpha_2 y_u - \beta_2 y_v\), in which

\[
(2.13) \quad \beta_2 = \left(\alpha^2 + 2a' + g + \alpha_v / 2a', \quad \alpha_2 = -\alpha - a' / 2a'.
\]

3. The edges of Green, the directrices of Wilczynski, and the projective normal. Let us apply the results of §§1 and 2 to obtain new characterizations of the edges of Green, the directrices of Wilczynski, and the projective normal of Fubini. The plane which is tangent to the cone (2.7) of the \(p\)-tangeodesics along the \(u\)-edge intersects the plane which is tangent to the cone (2.8) of the \(\sigma\)-tangeodesics along the \(v\)-edge in a line \(V\) of the second kind which will be called the joint-edge of the systems of \(p\)- and \(\sigma\)-tangeodesics of \(S\) at \(P_y\). This line passes through the points \(P_y\) and \(z\) where the general coordinates of \(z\) are given by \(z = y_{uv} - \alpha y_u - \beta y_v\), in which

\[
(3.1) \quad \alpha = -\alpha - a' / 2a', \quad \beta = -\beta - b_u / 2b.
\]

Since the functions \(\alpha, \beta\) associated with the edges of Green are given by

\[
(3.2) \quad \alpha = -a' / 4a', \quad \beta = -b_u / 4b,
\]

we have the following theorem.

**Theorem 3.1.** The second edge of Green at \(P_y\) lies in the plane \(\pi\) determined by the joint-edge of the systems of \(p\)- and \(\sigma\)-tangeodesics of \(S\) at \(P_y\) and the reciprocal \(l'\) of the line \(l\) joining \(p, \sigma\). The joint-edge coincides with the line \(l'\) if and only if \(l'\) is the second edge of Green. Any two particular planes \(\pi_1\) and \(\pi_2\) of the plane \(\pi\) (corresponding to selections \(\rho_1, \sigma_1\) and \(\rho_2, \sigma_2\)) intersect in the second edge of Green.

Let \(\sigma_\alpha\) denote the intersection of the tangent plane to \(S_p\) at \(p\) with the \(v\)-tangent to \(S\) at \(P_y\) and let \(\rho_\alpha\) denote the intersection of the tangent plane to \(S_\sigma\) at \(\sigma\) with the \(u\)-tangent to \(S\) at \(P_y\). It may be easily verified that the general coordinates of \(\rho_\alpha\) and \(\sigma_\alpha\) are given by

\[
(3.3) \quad \beta_\alpha = -\left(\alpha + \alpha_\alpha + \alpha^2\right) / 2\alpha', \quad \alpha_\alpha = -\left(f + \alpha_u + \beta^2\right) / 2b.
\]

The line \(l_\alpha\) joining \(\rho_\alpha, \sigma_\alpha\) was introduced by the author in a previous paper\(^2\) and called the asymptotic associate of the line \(l\) joining \(\rho, \sigma\).

The plane determined by the \(v\)-tangent of \(S\) at \(P_y\) and the \(u\)-edge of the \(p\)-tangeodesics at \(P_y\) is the polar plane of the \(u\)-tangent of \(S\) at \(P_y\) with respect to the cone (2.7) of the \(p\)-tangeodesics. Similarly, the

plane determined by the $u$-tangent of $S$ at $P_y$ and the $v$-edge of the 
$\sigma$-tangeodesics at $P_y$ is the polar plane of the $v$-tangent of $S$ at $P_y$
with respect to the cone (2.8) of the $\sigma$-tangeodesics. These two polar
planes intersect in a line $l_\delta$ which will be called the polar-axis of the
$\rho$- and $\sigma$-tangeodesics at $P_y$. This line may be shown to pass through
the points $P_y$ and $z_3$ where the general coordinates of $z_3$ are given by
$z_3 = y_{uv} - \alpha_3 y_u - \beta_3 y_v$, in which

$$(3.4) \quad \alpha_3 = -\alpha_a + b_v/b, \quad \beta_3 = -\beta_a + a_u'/a'.$$

Since the functions $\alpha, \beta$ for the directrix $l'$ of Wilczynski are given by

$$(3.5) \quad \alpha = b_v/2b, \quad \beta = a_u'/2a'.$$

equations (3.4) are such that we have, immediately, this theorem.

**Theorem 3.2.** The second directrix of Wilczynski lies in the plane
$p$ determined by the polar-axis of the $\rho$- and $\sigma$-tangeodesics at $P_y$ and the
reciprocal $l'_\delta$, with respect to $S$ at $P_y$, of the asymptotic associate of $l$. Any
two particular planes $p_1$ and $p_2$ of the plane $p$ (corresponding to lines
$l_1$ and $l_2$) intersect in the second directrix of Wilczynski.

Theorems 3.1 and 3.2 may be dualized by replacing the lines
and planes of these theorems by their reciprocals with respect to $S$ at $P_y$. The dual of Theorem 3.1 is the following theorem.

**Theorem 3.3.** The first edge of Green contains the point $P$ of inter­
section of the reciprocal of the joint-edge of the systems of $\rho$- and $\sigma$-tangeo­
desics of $S$ and the line $l$ joining $P, \sigma$. These three lines coincide if and
only if the line $l$ is the first edge of Green. Any two particular points
$P_1$ and $P_2$ of the point $P$ (corresponding to selections $\rho_1, \sigma_1$ and $\rho_2, \sigma_2$)
determine the first edge of Green.

The statement of the dual of Theorem 3.2 will be left to the care
of the reader.

Finally, since the projective-normal of $S$ at $P_y$ is the line for which the
functions $\alpha, \beta$ are given by $\alpha = -(b_v/2b + a_u'/2a')$, $\beta = -(a_u'/2a' + b_u/2b)$,
and the first directrix of $S$ at $P_y$ is the line $l$ for which $\alpha = b_v/2b$,
$\beta = a_u'/2a'$, we have from equations (3.1) this theorem.

**Theorem 3.4.** If the line $l$ joining $\rho, \sigma$ is the first directrix of
Wilczynski, the joint-edge of the systems of $\rho$ and $\sigma$-tangeodesics of $S$ at $P_y$
is the projective normal of Fubini.

**University of Kansas**