remarked that Theorem A may well carry, in such a study, a weight greater than that indicated by its relatively minor role in the proof of Theorem B.

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THE EQUIVALENCE OF \( n \)-MEASURE AND LEBESGUE MEASURE IN \( E^n \)

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Consider a set \( A \) of points in euclidean \( n \)-space \( E^n \). For each countable covering \( \{A_i\} \) of \( A \) by arbitrary sets consider the sum

\[
\sigma = \sum_i c_m \delta(A_i)^m,
\]

where \( m \) is a fixed positive number, \( c_m = \pi^{m/2}/2^m \Gamma((m+2)/2) \), and \( \delta(A) \) is the diameter of \( A \). The constant \( c_m \) is, for integral \( m \), the \( m \)-volume of a sphere of unit diameter in \( E^m \). Let \( L_m(A; \alpha) \) be the greatest lower bound of all sums \( \sigma \) corresponding to coverings for which \( \delta(A_i) < \alpha \) for all \( i \) \((\alpha > 0)\). We define the \( m \)-measure of \( A \) as \( L_m(A) = \lim_{\alpha \to 0} L_m(A; \alpha) \). We denote the outer Lebesgue measure of \( A \) by \(|A|\).

We shall show that \( n \)-measure and outer Lebesgue measure are equal: \( L_n(A) = |A| \). A statement on this matter by W. Hurewicz and H. Wallman is true but misleading: these authors assert that \( L_n(A)/c_n \) and \(|A|\) may be unequal.\(^1\)

F. Hausdorff has introduced an \( m \)-measure \( L^S_m(A) \) defined as is \( L_m(A) \) except that coverings by spheres are used instead of coverings by arbitrary sets. He has shown\(^2\) that \( L^S_m(A) = |A| \). However \( L_m(A) \) and \( L^S_m(A) \) are unequal in general, as A. S. Besicovitch has shown\(^3\) for \( m = 1, n = 2 \). S. Saks\(^4\) and others define \( m \)-measure as \( L_m(A)/c_m \).

Our proof, which is an obvious extension of Hausdorff's proof, depends on two known theorems.

THEOREM I. Of all sets in \( E^n \) having a given diameter, the \( n \)-sphere has the greatest outer Lebesgue measure.\(^5\)

\(^1\) W. Hurewicz and H. Wallman, Dimension theory, Princeton, 1941, p. 104.
\(^4\) S. Saks, Theory of the integral, Warsaw, 1937, pp. 53-54.
Theorem II. Suppose that to each point $x$ of a set $A$ in $E^n$ there corresponds a set of closed $n$-spheres centered at $x$ of arbitrarily small positive diameter. Then for any given $\varepsilon > 0$, a countable number of the spheres cover $A$ and are such that the sum of their Lebesgue measures is at most $|A| + \varepsilon$.

We now prove that

$$|A| \leq L_n(A) \leq L_n^S(A) \leq |A|.$$

For any countable covering $\{A_i\}$ of $A$,

$$|A| \leq \sum_i |A_i| \leq \sum_i c_n \delta(A_i)^n$$

by Theorem I. Hence $|A| \leq L_n(A ; \alpha)$ for all $\alpha$ and $|A| \leq L_n(A)$.

The definitions imply that $L_n(A) \leq L_n^S(A)$.

Finally, given $\varepsilon > 0$ and $\alpha > 0$, assign to each point $x$ of $A$ the set of all closed spheres centered at $x$ and of positive diameter less than $\alpha$. Then by Theorem II a countable number of these spheres $\{S_i\}$ cover $A$ and are such that

$$\sum_i |S_i| = \sum_i c_n \delta(S_i)^n \leq |A| + \varepsilon.$$

Hence $L_n^S(A ; \alpha) \leq |A|$ and $L_n^S(A) \leq |A|$.

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H. Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik vol. 27 (1916) p. 190. The case $|A| = \infty$ is not excluded.