ON THE CONVERGENCE OF CERTAIN PARTIAL SUMS OF A TAYLOR SERIES WITH GAPS

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We consider the function $f(z)$ determined by the power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

and its direct analytic continuation. For simplicity, it is supposed that $\lim \sup |c_n|^{1/n} = 1$.

We write

$$S_n(z) = \sum_{k=0}^{n} c_k z^k, \quad M(r) = \max_{|z|=r} |f(z)| \quad (0 < r < 1),$$

$$M(r) = 1 \quad (r \leq 0),$$

$$\theta_n = \frac{\lambda_{n+1}}{\lambda_n} - 1.$$

Ostrowski has proved\(^1\) that if $\{\theta_{n_i}\}$ is a sequence extracted from the sequence $\{\theta_n\}$ such that $\lim \inf \theta_{n_i} > 0$, then every regular point of $f(z)$ on the circle $|z| = 1$ is the center of a circle in which the sequence $\{S_{n_i}(z)\}$ converges uniformly to $f(z)$. Restricting ourselves to the question of convergence at the regular points themselves, we shall prove the following theorem:

**If**

$$\limsup_{i \to \infty} \frac{\log (M(1 - \theta_{n_i}^2)/\theta_{n_i})}{\lambda_n \theta_{n_i}^2} < \infty,$$

**then** $\lim S_{n_i}(z) = f(z)$ **at all regular points of** (1) **on the circle** $|z| = 1$.

For the proof, we shall assume that $\lim \theta_{n_i} = 0$; afterwards, we shall remove this restriction, with the aid of Ostrowski's theorem.

Let $z_1$ be a regular point for (1) on the circle $|z| = 1$, and let $z_0$ be a point on the segment joining $z_1$ to the origin. We write $|z_1 - z_0| = a$, and for every positive integer $i$ we define the three circles

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where \( \rho'' \) shall be chosen so that the function \( f(z) \) is holomorphic in the closed region bounded by \( \Gamma'' \). By \( M'(a), M'_i(a), \) and \( M'' \) we denote the maximum value of \( |f(z) - S_n(z)| \) on \( \Gamma', \Gamma(a), \) and \( \Gamma'' \), respectively. We write

\[
R'_i = 1 + \rho'_i - a, \quad R''_i = 1 + \rho'' - a,
\]
and we choose \( r_i \) so that \( R'_i < r_i < 1 \).

By Cauchy's formula we have, on the circle \( |z| = R'_i \),

\[
|f(z) - S_n(z)| \leq \sum_{\rho = 0}^{\infty} \frac{M(r_i)r_i}{r_i^p} = \frac{M(r_i)r_i}{r_i - R'_i} \left( \frac{R'_i}{r_i} \right)^{\lambda_{n+1}}.
\]
and by the principle of the maximum

\[
\log M'_i \leq \log \frac{M(r_i)r_i}{r_i - R'_i} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R'_i}{r_i}.
\]

On the circle \( |z| = R''_i \), and therefore also on \( \Gamma'' \), we have

\[
|S_n(z)| \leq M(r_i) \sum_{\rho = 0}^{\infty} \left( \frac{R''_i}{r_i} \right)^p < \frac{M(r_i)r_i}{R''_i - r_i} \left( \frac{R''_i}{r_i} \right)^{\lambda_{n_i}}.
\]
Since \( f(z) \) is holomorphic in the closed region bounded by \( \Gamma'' \), and since at least one of the expressions \( R''_i/(R''_i - r_i) \), \( (R''_i/r_i)^{\lambda_{n_i}} \) tends to \( \infty \) as \( i \to \infty \), we may write, for any positive \( \eta \) and for \( i \) sufficiently large,

\[
\log M'' \leq \log \frac{M(r_i)(1 + \eta)R''_i}{R''_i - r_i} + \lambda_{n_i} \log \frac{R''_i}{r_i}.
\]

Applying Hadamard's three-circle theorem to the function \( S_n(z) - f(z) \) on the circles \( \Gamma', \Gamma(a), \Gamma'' \), we have now

\[
\log \frac{R''_i}{R'_i} \log M_i(a) \leq \log \left( \frac{R''_i}{R'_i} \right) \log M'_i + \log \left( \frac{a}{\rho'_i} \right) \log M''.
\]

(3)

In the proof of his theorem Ostrowski now takes the point \( z_0 \) near
to the point \( z_1 \) and chooses for \( \Gamma', \Gamma(a) \), and \( \Gamma'' \) three fixed circles with radii sufficiently near to \( |z_1 - z_0| \) (with \( \Gamma(a) \) including the point \( z_1 \) instead of passing through it). In our case it is necessary to take \( z_0 \) near to the origin and to let \( \rho' \) and \( \rho'' \) tend to \( a \) as \( i \) becomes large. We choose \( \rho' = (1 - b_i)a, \rho'' = (1 + b_i)a, r_i = 1 - k_i \), where \( 0 < b_i < 1 \) and \( 0 < k_i < ab_i \); substituting these values in (3), expanding terms such as \( \log (1 + ab_i) \) in power series, and dividing both sides of the inequality by \( b_i \), we get

\[
2(1 + b_i^2/3 + \cdots ) \log M_i(a) < (1 - b_i/2 + b_i^2/3 - \cdots ) \\
\cdot \left\{ \log \frac{M(1 - k_i)(1 - k_i)}{(1 - k_i/ab_i)ab_i} + \lambda_n a \log \frac{1 - ab_i}{1 - k_i} \right\} \\
+ (1 + b_i/2 + b_i^2/3 + \cdots ) \\
\cdot \left\{ \log \frac{M(1 - k_i)(1 - k_i)(1 + \eta)}{(1 + k_i/ab_i)ab_i} + \lambda_n a \log \frac{1 + ab_i}{1 - k_i} \right\} \\
< 2\eta + 3k_i + 3k_i/ab_i - 3 \log a \\
b + 2(1 + b_i^2/3 + \cdots ) \log \frac{M(1 - k_i)}{b_i} \\
+ \lambda_n \left\{ \theta_n a \left[ 1 - b_i/2 + b_i^2/3 - \cdots \right] \\
\cdot \left[ - ab_i - a^2 b_i^2/2 - \cdots + k_i + k_i^2/2 + \cdots \right] \\
+ \left[ 1 - b_i/2 + b_i^2/3 - \cdots \right] [- ab_i - a^2 b_i^2/2 - \cdots ] \\
+ \left[ 1 + b_i/2 + b_i^2/3 + \cdots \right] [ab_i - a^2 b_i^2/2 + \cdots ] \\
+ 2\left[ 1 + b_i^2/3 + \cdots \right] [k_i + k_i^2/2 + \cdots ] \right\} \\
< 2\eta + 3k_i + 3k_i/ab_i - 3 \log a + 3 \log \frac{M(1 - k_i)}{b_i} \\
- \left( \lambda_n/b \right) \left\{ ab_i \left[ \theta_n - b_i(1 - a) \right] - 3k_i \right\}
\]

provided \( \theta_n \) and \( b_i \) are sufficiently small and the sum of the terms in the braces of the last term is positive.

Now suppose that

\[
\limsup_{i \to \infty} \frac{\log (M(1 - \theta_n^2)/\theta_n)}{\lambda_n \theta_n^2} < H \quad (1 < H < \infty).
\]

Choose \( b_i = b \theta_n, k_i = b_i \theta_n/b = \theta_n^2, \) where \( b > 14H \). The last member of (4) becomes

\[
- \left( \lambda_n \theta_n^2 b/2 \right) \left\{ a \left[ 1 - b(1 - a) \right] - 3/b \right\}.
\]
If \( a \) is chosen sufficiently near to 1, \( b \{a[1-b(1-a)]-3/b\} > 10H, \) and for sufficiently large values of \( i \) (4) becomes

\[
(2 + \epsilon) \log M_i(a) < K(a, b) + 3 \log M(1 - \theta_{n_i}/\theta_{n_i} - 5\lambda_n\theta_{n_i} H,
\]
where \( K(a, b) \) depends on \( a \) and \( b \) only. But \( \lim_{i \to \infty} \theta_{n_i} = 0 \) implies, together with the validity of (2), that \( \lim_{i \to \infty} \lambda_n \theta_{n_i}^2 = \infty \). For sufficiently large \( i \) we have

\[
(2 + \epsilon) \log M_i(a) < -\lambda_n\theta_{n_i}^2 H,
\]
that is,

\[
\lim_{i \to \infty} \log M_i(a) = -\infty,
\]
and, in particular,

\[
\lim_{i \to \infty} S_{n_i}(z_1) = f(z_1).
\]

Now let \( \{\theta_{n_i}\} \) be any sequence of values \( \theta_{n_i} \) for which (2) is satisfied, and let \( z_1 \) be a regular point for (1). From every subsequence of \( \{\theta_{n_i}\} \) we can extract a further subsequence \( \{\theta_{m_j}\} \) such that either \( \lim \theta_{m_j} = 0 \) or \( \lim \inf \theta_{m_j} > 0 \). In the first case, \( \lim S_{m_j}(z_1) = f(z_1) \) by what we have just proved; in the second case, by Ostrowski’s theorem. From every subsequence of \( \{n_i\} \) we can therefore extract a further subsequence \( \{m_j\} \) such that \( \lim S_{m_j}(z_1) = f(z_1) \). It follows that the sequence \( \{S_{n_i}(z_1)\} \) itself tends to \( f(z_1) \), and our theorem is proved.

Condition (2) may be replaced by one that is somewhat less general, but can be expressed more immediately in terms of the behavior of \( M(r) \):

If \( \lim \inf \theta_{n_i} = \theta \), we define, for \( h > \theta \),

\[
\lambda(h) = \text{g.l.b.} \lambda_{n_i};
\]
for \( 0 \leq h \leq \theta \), we write \( \lambda(h) = \infty \). It follows from our theorem that \( \lim S_{n_i}(z_1) = f(z_1) \) whenever \( z_1 \) is a regular point for (1) on the circle \( |z| = 1 \), provided

\[
\limsup_{h \to 0+} \frac{\log \left( M(1 - h^2)/h \right)}{h^2 \lambda(h)} < \infty.
\]

If we write \( r = 1 - h^2 \), (5) becomes

\[
\limsup_{r \to 1-0} \frac{\log \left[ M(r)/(1 - r)^{1/2} \right]}{(1 - r)\lambda(1 - r)^{1/2}} < \infty.
\]
ON ABEL AND LEBESGUE SUMMABILITY

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1. Introduction. A series \( \sum a_n \) is called Abel summable to the value \( s \) if the power series \( \sum a_n r^n \) converges for \( 0 < r < 1 \), and if \( \sum a_n r^n = s \) as \( r \uparrow 1 \); it is called Lebesgue summable if the sine series

\[
\sum_1^\infty a_n \frac{\sin nt}{n} = F(t)
\]

converges in some interval \( 0 < t < \tau \), and if

\[
t^{-1}F(t) \to s \quad \text{as} \quad t \downarrow 0.
\]

We write in the first case \( A \sum a_n = s \), and in the latter case \( L \sum a_n = s \) (summability \( A \) or \( L \) respectively). It is known that convergence does not imply \( A \)-summability and conversely \( L \)-summability does not imply convergence of \( \sum a_n \). Tauberian type problems which arise out of this situation have been discussed.\(^1\) It is also known that either convergence or \( L \)-summability imply \( A \)-summability. As to the converse (restricting ourselves to real \( a_n \)) we have proved the following theorems:

Theorem 1. [8, pp. 582–583]. If

\[
\sum_{n=1}^{2n} \left( | a_n | - a_n \right) = O(1) \quad \text{as} \quad n \to \infty,
\]

and if

\[
\sum_{n=1}^{\infty} a_n r^n = O(1) \quad \text{as} \quad r \uparrow 1,
\]

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\(^1\) See [8], where further references are given; numbers in brackets refer to the bibliography at the end of this paper.