versely if the first row is multiplied by the inverse of \( c \) (mod \( p^k \)). This inverse exists, and the correspondence is one-to-one, because \( c \) is prime to \( p \). This proves (3).

The sum of the probabilities \( P_n(a^p^a, p^k) \), where \( a \) runs through the values \( 1, 2, \ldots, p^{k-a} \), is clearly the probability that a determinant be divisible by \( p^a \). The terms of this sum can be simplified and collected by use of (3), and we have

\[
P_n(0, p^a) = \sum_{r=0}^{k-a} \phi(p^{k-a-r})P_n(p^{a+r}, p^k).
\]

Replacing \( a \) by \( a+1 \), and subtracting the resulting equation from (11), we arrive at (4).

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ON THE NOTION OF THE RING OF QUOTIENTS OF A PRIME IDEAL

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Let \( \mathfrak{o} \) be a domain of integrity (that is, a ring with unit element and with no zero divisor not equal to 0), and let \( u \) be a prime ideal in \( \mathfrak{o} \). We can construct two auxiliary rings associated with \( u \): the factor ring \( \mathfrak{o}/u \), composed of the residue classes of elements of \( \mathfrak{o} \) modulo \( u \), and the ring of quotients \( \mathfrak{o}_u \), composed of the fractions whose numerator and denominator belong to \( \mathfrak{o} \), but whose denominators do not belong to \( u \). These constructions are of paramount importance in algebraic geometry; if \( \mathfrak{o} \) is the ring of a variety \( \mathcal{V} \), there corresponds to \( u \) a subvariety \( \mathcal{U} \) of \( \mathcal{V} \); \( \mathfrak{o}/u \) is the ring of \( \mathcal{U} \), whereas the ring \( \mathfrak{o}_u \) is the proper algebraic tool to investigate the neighborhood of \( \mathcal{U} \) with respect to \( \mathcal{V} \).

Now, the local theory of algebraic varieties involves the consideration of rings which are not domains of integrity (this, because the completion of a local ring may introduce zero divisors). Let then \( \mathfrak{o} \) be any commutative ring with unit element, and let again \( u \) be a prime ideal in \( \mathfrak{o} \). We may define the factor ring \( \mathfrak{o}/u \) exactly in the same way as above, but we cannot so easily generalize the notion of the ring of quotients \( \mathfrak{o}_u \). If there exist zero divisors outside \( u \), these zero divisors cannot be used as denominators of fractions, which shows that the definition of \( \mathfrak{o}_u \) cannot be extended verbatim. If we

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consider only fractions whose denominators are not zero divisors and
do not belong to \( \mathfrak{u} \), we obtain a ring \( \mathfrak{o}' \); but \( \mathfrak{o}' \) fails in general to have
the essential property of a ring of quotients, namely, of being a local
ring in the sense of Krull (that is, the non-units in \( \mathfrak{o}' \) will not form an
ideal). The object of this note is to construct a ring for which the
essential properties of rings of quotients are preserved.

Throughout this paper we shall denote by \( \mathfrak{o} \) a Noetherian ring
(that is, a ring in which the maximal condition for ideals is satisfied)
with a unit element. Generalizing the problem of defining the ring
of quotients of a prime ideal, we take any multiplicatively closed
subset \( S \) of \( \mathfrak{o} \) which does not contain 0 (a set is said to be multiplica-
tively closed if the product of any two elements of the set belongs to
the set; if we are concerned with a prime ideal \( \mathfrak{u} \) in \( \mathfrak{o} \), we take \( S \)
to be the complement of \( \mathfrak{u} \) in \( \mathfrak{o} \)). There exists at least one primary ideal
which does not meet \( S \) (otherwise, 0 would belong to \( S \) as we see at
once by representing the zero ideal as an intersection of primary
ideals). We shall denote by \( \mathfrak{s} \) the intersection of all primary ideals
in \( \mathfrak{o} \) which do not meet \( S \).

**Proposition 1.** Let \( \{0\} = q_1 \cap \cdots \cap q_h \) be an irredundant representa-
tion of the zero ideal in \( \mathfrak{o} \) as an intersection of primary ideals, and let
\( \mathfrak{p}_i (1 \leq i \leq h) \) be the associated prime ideal of \( q_i \). Assume that \( \mathfrak{p}_i \cap S = \emptyset \)
for \( i \leq g \), but not for \( i > g \). Then \( \mathfrak{s} = q_1 \cap \cdots \cap q_g \).

It is clear that \( \mathfrak{s} \subset q' = q_1 \cap \cdots \cap q_g \). Let \( \mathfrak{b} \) be any primary ideal
which does not meet \( S \); we shall prove that \( q' \subset \mathfrak{b} \). Let \( q'' \) be the ideal \( q_2 \cap \cdots \cap q_h \). We have \( \{0\} = q' \cap q'' = q'q'' \subset \mathfrak{b} \), whence \( q' \subset \mathfrak{b} : q'' \). Let
\( \mathfrak{u} \) be the associated prime ideal of \( \mathfrak{b} \); since \( \mathfrak{b} \) contains some power of \( \mathfrak{u} \),
it follows from the multiplicatively closed character of \( S \) that \( \mathfrak{u} \) does
not meet \( S \). If \( i > g \), the ideal \( \mathfrak{p}_i \) meets \( S \) and is therefore not contained
in \( \mathfrak{u} \). It follows\(^2\) that \( \mathfrak{b} : q'' = \mathfrak{b} \), whence \( q' \subset \mathfrak{b} \). Proposition 1 is thereby
proved.

**Lemma 1.** Let \( \mathfrak{p} \) be a prime ideal in \( \mathfrak{o} \), and let \( \mathfrak{a} \) be an ideal contained
in \( \mathfrak{p} \). If \( q \) is an ideal containing \( \mathfrak{a} \), the statements "\( q \) is primary for \( \mathfrak{p} \)"
and "\( q/\mathfrak{a} \) is primary for \( \mathfrak{p}/\mathfrak{a} \)" are equivalent.

\(^1\) It was H. Grell who observed for the first time that, \( S \) being any multiplicatively
closed set of nonzero divisors in a ring, it is possible to associate with \( S \) a ring of
quotients, whose elements are the fractions whose denominators belong to \( S \) (Cf.
p. 510). For the properties of these rings of quotients, cf. Krull, *Idealtheorie* (Ergeb-
nisse der Mathematik) or my paper *On the theory of local rings*, Ann. of Math. vol. 44
(1943) p. 690.

\(^2\) Cf. van der Waerden, *Moderne Algebra*, vol. 2, chap. 12, p. 36.
Lemma 1 follows trivially from the definitions.

The zero ideal in \( o/\mathfrak{g} \) is the intersection of the primary ideals \( \mathfrak{q}_1/\mathfrak{g}, \ldots, \mathfrak{q}_n/\mathfrak{g} \) whose associated prime ideals are \( \mathfrak{p}_1, \ldots, \mathfrak{p}_n \). Let \( S^* \) be the set of the residue classes modulo \( \mathfrak{g} \) of the elements of \( S \); then \( S^* \cap (\mathfrak{p}_i/\mathfrak{g}) = \emptyset \) \( (1 \leq i \leq n) \), which means that no element of \( S^* \) is a zero divisor in \( o/\mathfrak{g} \). We may therefore construct the ring of quotients \( (o/\mathfrak{g})_{S^*} \) of \( S^* \) with respect to the ring \( o/\mathfrak{g} \).

**Definition 1.** The ring \( (o/\mathfrak{g})_{S^*} \) will be called the ring of quotients of \( S \) with respect to \( o \). This ring will be denoted by \( o_{S^*} \).

This definition coincides with the usual one in the case where \( S \) does not contain any zero divisor. We shall now prove that the essential properties of rings of quotients in the usual sense still hold in our case.

If \( a \) is an ideal in \( o \), \( (a+\mathfrak{g})o_S \) is an ideal in \( o_S \) which we shall denote symbolically by \( a_{o_S} \) (in spite of the fact that \( o \) is not in general a subring of \( o_S \), so that we cannot multiply elements of \( o \) by elements of \( o_S \)). If \( b \) is any ideal in \( o_S \), the set \( b \cap (o/\mathfrak{g}) \) may be written in the form \( a/\mathfrak{g} \), where \( a \) is an ideal in \( o \) which contains \( \mathfrak{g} \). We shall denote \( a \) symbolically by \( b_{\cap o} \) (although \( b_{\cap o} \) is not a set theoretic intersection).

**Proposition 2.** If \( b \) is any ideal in \( o_S \), we have \( b = (b_{\cap o})o_S \).

Since \( o_S \) is a ring of quotients of \( o/\mathfrak{g} \), we have \( b = (b_{\cap (o/\mathfrak{g})})o_S \).

Thus Proposition 2 follows immediately from this formula.

**Proposition 3.** Let \( \mathfrak{p} \) be a prime ideal in \( o \), and let \( \mathfrak{q} \) be primary for \( \mathfrak{p} \). If \( \mathfrak{q} \) meets \( S \), we have \( \mathfrak{p}o_S = \mathfrak{q}o_S = o_S \). If \( \mathfrak{p} \) does not meet \( S \), \( \mathfrak{p}o_S \) is prime, \( \mathfrak{q}o_S \) is primary for \( \mathfrak{p}o_S \) and \( \mathfrak{p}o_S \cap o = \mathfrak{p} \), \( \mathfrak{q}o_S \cap o = \mathfrak{q} \).

If \( \mathfrak{q} \) meets \( S \), \( \mathfrak{q}+\mathfrak{g}/\mathfrak{g} \) meets \( S^* \), whence \( \mathfrak{p}o_S = \mathfrak{q}o_S = o_S \). If \( \mathfrak{p} \) does not meet \( S \), the same holds for \( \mathfrak{q} \), whence \( \mathfrak{g} \subset \mathfrak{q} \subset \mathfrak{p} \). By Lemma 1, \( \mathfrak{p}/\mathfrak{g} \) is prime and \( \mathfrak{q}/\mathfrak{g} \) is primary for \( \mathfrak{p}/\mathfrak{g} \). Furthermore, \( \mathfrak{p}/\mathfrak{g} \) does not meet \( S^* \). Proposition 3 follows therefore from the corresponding proposition which is known to hold for ordinary rings of quotients.

We see also that, if \( \mathfrak{p} \) does not meet \( \mathfrak{g} \), the formula \( \mathfrak{q} \leftrightarrow \mathfrak{q}o_S \) establishes a one-to-one inclusion preserving correspondence between the primary ideals for \( \mathfrak{p} \) in \( o \) and the primary ideals for \( \mathfrak{p}o_S \) in \( o_S \).

**Proposition 4.** Let \( a = \mathfrak{v}_1 \cap \cdots \cap \mathfrak{v}_r \) be an irredundant representation of an ideal \( a \) in \( o \) as an intersection of primary ideals. Let \( \mathfrak{u}_i \) be the associated prime ideal of \( \mathfrak{v}_i \), and assume that \( \mathfrak{u}_i \) meets \( S \) for \( i > s \) but not for \( i \leq s \). Then we have \( \mathfrak{a}o_S = \mathfrak{v}_1 o_S \cap \cdots \cap o_S \mathfrak{v}_r \mathfrak{a}o_S \), and this is an irredundant representation of \( \mathfrak{a}o_S \) as intersection of primary ideals in \( o_S \).

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1 If \( s^* \subseteq S^* \), we have \( \{0\} : s^*(o/\mathfrak{g}) = \{0\} \) by the result quoted in footnote 2.
It is obvious that \( \mathfrak{a}_0 \mathfrak{s} \subseteq \mathfrak{v}_0 \mathfrak{s} \cap \cdots \cap \mathfrak{v}_n \mathfrak{s} \). Let conversely \( a \) be any element of \( \mathfrak{v}_0 \mathfrak{s} \cap \cdots \cap \mathfrak{v}_n \mathfrak{s} \). We know that \( \mathfrak{v}_0 \mathfrak{s} \cap \cdots \cap \mathfrak{v}_n \mathfrak{s} \) is equal to \( (\mathfrak{b}_1 / \mathfrak{s} \cap \cdots \cap \mathfrak{b}_n / \mathfrak{s})_{\mathfrak{a}_0 \mathfrak{s}} \); it follows that \( a \) may be written in the form \( b^* / c^* \), where \( b^* \subseteq \mathfrak{b}_1 / \mathfrak{s} \cap \cdots \cap \mathfrak{b}_n / \mathfrak{s} \) and \( c^* \subseteq \mathfrak{S}^* \). If \( i > s \), the ideal \( \mathfrak{u}_i \) has an element \( u_1 \) in common with \( \mathfrak{S} \); if \( m \) is large enough, we have \( u = (\prod_{i=s+1}^{m} u_i)^m \subseteq \mathfrak{v}_{s+1} \cap \cdots \cap \mathfrak{v}_n \). Since \( u_i \subseteq \mathfrak{S} \) \( (s+1 \leq i \leq r) \), we have \( u \subseteq \mathfrak{S} \), whence \( u^* \subseteq \mathfrak{S}^* \), if \( u^* \) is the residue class of \( u \) modulo \( \mathfrak{s} \). We may write \( a = (u^* b^*) / (u^* c^*) \), \( u^* c^* \subseteq \mathfrak{S}^* \). Let \( b \) be any element of the residue class \( b^* \) modulo \( \mathfrak{s} \); since \( b^* \subseteq \mathfrak{b}_1 / \mathfrak{s} \) \( (1 \leq i \leq s) \), we have \( b \subseteq \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_n \), whence \( \mathfrak{b} b \subseteq \mathfrak{a} \) and \( u^* b^* \subseteq \mathfrak{a} + \mathfrak{s} \), \( a \subseteq \mathfrak{a}_0 \mathfrak{s} \). It is clear that none of the ideals \( \mathfrak{v}_i / \mathfrak{s} \) contains the intersection of the others; making use of a known result\(^4\) for ordinary rings of quotients, it follows that the representation \( \mathfrak{a}_0 \mathfrak{s} = \mathfrak{v}_0 \mathfrak{s} \cap \cdots \cap \mathfrak{v}_n \mathfrak{s} \) is irredundant.

We shall now consider more specifically the case where \( \mathfrak{S} \) is the complement of a prime ideal \( \mathfrak{u} \). The ring \( \mathfrak{a}_0 \mathfrak{s} \) will then also be denoted by \( \mathfrak{a}_0 \mathfrak{u} \). In that case, the ideal \( \mathfrak{s} \) coincides with the intersection of all primary ideals for \( \mathfrak{u} \). In fact, the set \( \mathfrak{S}^* \) is clearly the complement of \( \mathfrak{u} / \mathfrak{s} \) with respect to \( \mathfrak{a} / \mathfrak{s} \); the ring \( \mathfrak{a}_0 \mathfrak{s} \) is the ring of quotients (in the ordinary sense) of the prime ideal \( \mathfrak{u} / \mathfrak{s} \) with respect to \( \mathfrak{a} / \mathfrak{s} \). It follows that \( \mathfrak{u}_0 \mathfrak{s} \) is the ideal of non-units in \( \mathfrak{a}_0 \mathfrak{s} \), whence \( \cap_{s-1}^\infty (\mathfrak{u} / \mathfrak{s})^* \mathfrak{a}_0 \mathfrak{s} = \{ 0 \} \).\(^4\)

For every \( n \), the ideal \( (\mathfrak{u} / \mathfrak{s})^* \mathfrak{a}_0 \mathfrak{s} \mathfrak{a}_0 / \mathfrak{s} \) is a primary ideal for \( \mathfrak{u} / \mathfrak{s} \) in \( \mathfrak{a} / \mathfrak{s} \); it follows that the intersection of all primary ideals for \( \mathfrak{u} / \mathfrak{s} \) is the zero ideal in \( \mathfrak{a} / \mathfrak{s} \). Our assertion then follows from Lemma 1.\(^5\) At the same time, we see that \( \mathfrak{a}_0 \mathfrak{u} \) is a local ring in the sense of Krull.

Assume now that \( \mathfrak{u} \) is a semi-local ring\(^6\) and that \( \mathfrak{u} \) is one of the maximal prime ideals in \( \mathfrak{a} \). Let \( \mathfrak{b} \) be the completion of \( \mathfrak{a} \); there corresponds to \( \mathfrak{u} \) an idempotent \( e \) in \( \mathfrak{b} \). We shall prove the following results:

**Proposition 5.** The ring \( \mathfrak{a} / \mathfrak{s} \) is isomorphic with the subring \( \mathfrak{a} e \) of the ring \( \mathfrak{a} e \). This isomorphism may be extended to an isomorphism of the completion of \( \mathfrak{a}_0 \mathfrak{u} \) with \( \mathfrak{a} e \).

The first statement will be proved if we show that \( \mathfrak{s} \) coincides with the set of elements \( x \subseteq 0 \) which satisfy the condition \( xe = 0 \). If \( x \) is any

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\(^5\) This result, together with Proposition 1 above, yields a proof of a theorem of Krull; cf. Krull, *Primidealketten in allgemeinen Ringbereichen*, Sitzungsberichte der Heidelberger Akademie, 1928, p. 7.

\(^6\) A semi-local ring is a Noetherian ring (that is, the maximal chain condition holds in the ring) in which there exist only a finite number of maximal prime ideals. For the proofs of the results on semi-local rings which are used in this paper, cf. my paper quoted in footnote 1.
element of \(\mathfrak{o}\), we may write \(x = x_\epsilon + x(1 - \epsilon)\), and we know that \(1 - \epsilon \in \mathfrak{u}^n\) for every \(n\). If \(x_\epsilon = 0\), we have \(x \in \mathfrak{u}^n\mathfrak{o} = \mathfrak{u}^n\) for every \(n\); in particular, \(x\) belongs to every primary ideal for \(\mathfrak{u}\), whence \(x \in \mathfrak{e}\).

If \(x \in \mathfrak{e}\), we have \(x \in \mathfrak{u}^n\) for every \(n\) (\(\mathfrak{u}^n\) is primary because \(\mathfrak{u}\) is a maximal prime ideal), whence \(x_\epsilon \in \mathfrak{u}^n\mathfrak{e}\). Since \(\mathfrak{u}\mathfrak{e}\) is the ideal of non-units in \(\mathfrak{e}\), it follows that \(x_\epsilon = 0\).

If we identify \(\mathfrak{o}/\mathfrak{e}\) with \(\mathfrak{o}\mathfrak{e}\), every element of \(S^*\) is a unit in \(\mathfrak{e}\). In fact, if \(y \in S\), we have \(y = y_\epsilon + y(1 - \epsilon)\), \(y(1 - \epsilon) \in \mathfrak{u}\). If we had \(y \in u\mathfrak{e}\), we would have \(y \in \mathfrak{u}\mathfrak{e} = \mathfrak{u}\), which is not the case. It follows that \(\mathfrak{e}\) contains the ring \(\mathfrak{o}\mathfrak{u}\). The ring \(\mathfrak{e}\) is a complete ring with \(\mathfrak{e}\mathfrak{u}\) as unique maximal prime ideal; it is clear that \(\mathfrak{e}\) (and, a fortiori, \(\mathfrak{o}\mathfrak{u}\)) is dense in \(\mathfrak{e}\). In order to prove that \(\mathfrak{e}\) is the completion of \(\mathfrak{o}\mathfrak{u}\), it is sufficient to prove that \(\mathfrak{o}\mathfrak{u}\) is topologically a subspace of \(\mathfrak{e}\). We show first that \(\mathfrak{u}^n\mathfrak{e} \cap \mathfrak{e} = \mathfrak{u}^n\mathfrak{e}\) for every \(n\). Let \(x_\epsilon (x \in \mathfrak{o})\) be an element of \(\mathfrak{u}^n\mathfrak{e} \cap \mathfrak{e}\); we have \(x = x_\epsilon + x(1 - \epsilon) \in \mathfrak{u}^n\), whence \(x \in \mathfrak{u}^n\mathfrak{e} \cap \mathfrak{e} = \mathfrak{u}^n\), \(x_\epsilon \in \mathfrak{u}^n\). The ideal \(\mathfrak{u}^n\mathfrak{e} \mathfrak{o}\mathfrak{u}\) is equal to \(((\mathfrak{u}^n\mathfrak{e} \mathfrak{o}\mathfrak{u}) \cap \mathfrak{e}) \mathfrak{o}\mathfrak{u} = (\mathfrak{u}^n\mathfrak{e}) \mathfrak{o}\mathfrak{u} = \mathfrak{u}^n\mathfrak{o}\mathfrak{u}\); Proposition 5 is thereby completely proved.

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