UNION-PRESERVING TRANSFORMATIONS OF SPACE

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1. **General statement.** Sophus Lie studied transformations from lineal-elements into lineal-elements, and also transformations from surface-elements into surface-elements of space. The contact group is obtained by requiring all unions to be converted into unions. Lie's fundamental theorems may be stated as follows. All the contact lineal-element transformations form the group of extended point transformations. The contact surface-element transformations which are not merely extended point transformations are defined completely by either a single directrix equation, or a pair of directrix equations. In the first case, a point corresponds to a surface; and in the second case, a point corresponds to a curve.[1]

We extend the preceding results by studying transformations in space from differential curve-elements of order n: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where n is 2 or more, into lineal-elements (X, Y, Z, Y', Z'). An example of such a transformation arises in the problem of finding the locus of the centers of spherical curvature for an arbitrary space curve. This problem leads to a transformation from curve-elements of third order into lineal-elements.[2]

We determine the general class of union-preserving transformations by means of a directrix equation. Lie has obtained directrix equations only for contact transformations of surface-elements since there are no contact transformations of lineal-elements besides the extended point transformations. For a point-to-surface transformation, Lie's standard directrix equation is of the form $\Omega(X, Y, Z, x, y, z) = 0$. For a point-to-curve transformation, there are two standard directrix equations of the forms $\Omega_1(X, Y, Z, x, y, z) = 0$, $\Omega_2(X, Y, Z, x, y, z) = 0$. We find that any general union-preserving transformation from curve-elements of order n into lineal-elements is completely determined by our *new directrix equation*, involving derivatives, $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)}) = 0$.

In the final part of our paper, we shall prove that the only available union-preserving transformations (in the whole domain of curve-

Presented to the Society, November 27, 1943; received by the editors August 17, 1943. We have studied the two-dimensional aspects of our new theory in Proc. Nat. Acad. Sci. (1943) and Revista des Matimaticas (1943). This leads to a large extension of the Huygens theory of evolutes and involutes and Lie theory.

[1] Lie-Scheffers, *Beruhrungstransformationen.*

[2] See Bull. Amer. Math. Soc. abstract 49-9-235 by Kasner and DeCicco, *A generalized theory of contact transformations.*

elements) are firstly, the group of point transformations, and secondly, the set of union-preserving transformations from curve-elements of order n into lineal-elements, together with the extensions of these two types.

2. **The union-preserving transformations.** Any transformation from curve-elements of order n, where n is 2 or more, into lineal-elements may be defined by equations of the form

$$\begin{aligned} X &= X(x, y, z, p_1, q_1, \cdots, p_n, q_n), \\ Y &= Y(x, y, z, p_1, q_1, \cdots, p_n, q_n), \\ Z &= Z(x, y, z, p_1, q_1, \cdots, p_n, q_n), \\ P &= P(x, y, z, p_1, q_1, \cdots, p_n, q_n), \\ Q &= Q(x, y, z, p_1, q_1, \cdots, p_n, q_n), \end{aligned} \tag{1}$$

where $p_m = y^{(m)} = d^m y/dx^m$ and $q_m = z^{(m)} = d^m z/dx^m$ for $m = 1, 2, \cdots, n$.

A series of curve-elements of order n may be defined by $y = y(x)$, $z = z(x)$, $p_1 = p_1(x)$, $q_1 = q_1(x)$, $\cdots$, $p_n = p_n(x)$, $q_n = q_n(x)$. A series is a union if and only if the conditions $dy - p_1 dx = 0$, $dz - q_1 dx = 0$, $\cdots$, $dp_{n-1} - p_n dx = 0$, $dq_{n-1} - q_n dx = 0$ are satisfied.

A special type of union is the conical-union of order n. This consists of ∞^1 curve-elements of order n which have in common a fixed curve-element of order $(n-1)$. The equations of any conical-union are $x = x_0$, $y = y_0$, $z = z_0$, $p_1 = (p_1)_0$, $q_1 = (q_1)_0$, $\cdots$, $p_{n-1} = (p_{n-1})_0$, $q_{n-1} = (q_{n-1})_0$, $q_n = F(p_n)$.

Let us now consider what unions become unions under the transformation (1). These unions must satisfy the pair of Monge differential equations of order $(n+1)$

$$\begin{aligned} &(Y_{p_n} - PX_{p_n})p_{n+1} + (Y_{q_n} - PX_{q_n})q_{n+1} \\ &\quad + [(Y_x + p_1 Y_y + q_1 Y_z + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}}) \\ &\quad - P(X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] = 0, \\ &(Z_{p_n} - QX_{p_n})p_{n+1} + (Z_{q_n} - QX_{q_n})q_{n+1} \\ &\quad + [(Z_x + p_1 Z_y + q_1 Z_z + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}}) \\ &\quad - Q(X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] = 0. \end{aligned} \tag{2}$$

THEOREM 1. *All transformations from curve-elements of order n into lineal-elements may be divided into three distinct types with respect to the number of unions which are converted into unions.*[3]

[3] Kasner, *General transformation theory of differential elements*, Amer. J. Math. vol. 32 (1904) pp. 392–401.

Type 1°. *The union-preserving transformations. These carry every union into a union. The necessary and sufficient conditions for a union-preserving transformation are*

$$
\begin{aligned}
P &= \frac{Y_x + p_1 Y_y + q_1 Y_z + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}}}{X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}}} \\
&= \frac{Y_{p_n}}{X_{p_n}} = \frac{Y_{q_n}}{X_{q_n}}, \\
Q &= \frac{Z_x + p_1 Z_y + q_1 Z_z + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}}}{X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-}}} \\
&= \frac{Z_{p_n}}{X_{p_n}} = \frac{Z_{q_n}}{X_{q_n}}.
\end{aligned} \tag{3}
$$

Type 2°. *The intermediate transformations. These are not union-preserving transformations and satisfy the conditions*

$$
\begin{aligned}
&[(Y_x + p_1 Y_y + q_1 Y_z + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}}) \\
&\qquad - P(X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] \\
&\cdot [(Z_x + p_1 Z_y + q_1 Z_z + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}}) \\
&\qquad - Q(X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})]^{-1} \\
&\qquad = \frac{Y_{p_n} - PX_{p_n}}{Z_{p_n} - QX_{p_n}} = \frac{Y_{q_n} - PX_{q_n}}{Z_{q_n} - QX_{q_n}}.
\end{aligned} \tag{4}
$$

Any intermediate transformation carries the ∞^∞ *unions of a Monge differential equation of third order into unions.*

Type 3°. *The general transformations. These do not satisfy the conditions* (4). *Any such transformation converts precisely* $\infty^{2(n+1)}$ *unions into unions.*

The general Type 3° is defined by *five* arbitrary functions of $(2n+3)$ independent variables. The intermediate Type 2° is defined by *three* arbitrary functions of $(2n+3)$ independent variables. For this intermediate type, it is seen that (X, Y, Z) may be taken arbitrarily and then the functions P and Q are completely determined by the equations (4). Later we shall show that the union-preserving Type 1° is defined by *one* arbitrary function of $(2n+1)$ variables or by *three* arbitrary functions of $(2n+1)$ variables.

We prove Theorem 1. For union-preserving transformations the equations (2) must be identities in p_{n+1} and q_{n+1}. Upon setting the various coefficients equal to zero and solving for P and Q, we obtain

the conditions (3) for union-preserving transformations.

Next suppose we have the intermediate transformations so that the conditions (4) are satisfied. Then it is obvious that the equations (2) are equivalent. This shows that under any intermediate transformation, there are ∞^{∞} unions preserved. These satisfy either one of the two equivalent Monge differential equations (2).

Finally suppose the conditions (4) are not satisfied. In the first place it is seen that if the last equality of (4) is not valid, we can solve the equations (2) for p_{n+1} and q_{n+1}. Therefore there are $\infty^{2(n+1)}$ unions which become unions. Finally if the last equality of (4) is valid, then it can be shown that at each curve-element of order $(n-1)$ there are ∞^{1} conical-unions which become unions. Since there are $\infty^{(2n+1)}$ curve-elements of order $(n-1)$, it follows that altogether there are $\infty^{2(n+1)}$ conical-unions which are converted into unions. Therefore we have proved that our general Type 3° carries exactly $\infty^{2(n+1)}$ unions into unions under all conditions.

3. **The degenerate union-preserving transformations.** Henceforth we shall consider only union-preserving transformations. However, there are certain degenerate union-preserving transformations which we wish to exclude from consideration. These degenerate correspondences are those where P and Q assume identically the values 0 or ∞; and also those for which there exist two functional relations between the three functions defininining (X, Y, Z).

It is found that these degenerate transformations may be divided into four cases: (a) those which convert every union into a union in a fixed plane parallel to one of the coordinate planes, (b) those which carry every union into a fixed straight line parallel to one of the coordinate axes, (c) those which carry every union into a conical-union of lineal-elements with fixed base-point, and (d) those which carry every union into a single fixed union.

In the rest of our article, whenever we speak of a union-preserving transformation we shall mean one which is not of the four preceding degenerate types.

4. **The special union-preserving transformations.** Any (nondegenerate) union-preserving transformation is said to be *special* if it carries every conical-union of order n into a conical-union of lineal-elements. Any such special transformation is given by equations of the form

$$
\begin{aligned}
X &= X(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),\\
Y &= Y(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),\\
Z &= Z(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),
\end{aligned}
\tag{5}
$$

$$(5)\qquad P=\frac{Y_x+p_1Y_y+q_1Y_z+\cdots+p_nY_{p_{n-1}}+q_nY_{q_{n-1}}}{X_x+p_1X_y+q_1X_z+\cdots+p_nX_{p_{n-1}}+q_nX_{q_{n-1}}},$$
$$Q=\frac{Z_x+p_1Z_y+q_1Z_z+\cdots+p_nZ_{p_{n-1}}+q_nZ_{q_{n-1}}}{X_x+p_1X_y+q_1X_z+\cdots+p_nX_{p_{n-1}}+q_nX_{q_{n-1}}}.$$

These special union-preserving transformations depend on three arbitrary functions of $(2n+1)$ independent variables. Thus there are $\infty^{3f(2n+1)}$ special union-preserving transformations.

5. **The directrix equation of a general union-preserving transformation** T. We shall say that a union-preserving transformation T is *general* if it does not carry every conical-union of order n into a conical-union of lineal-elements. It follows that for such a general transformation T, we must have either $X_{p_n}\neq 0$, $Y_{p_n}\neq 0$, $Z_{p_n}\neq 0$, or $X_{q_n}\neq 0$, $Y_{q_n}\neq 0$, $Z_{q_n}\neq 0$, or both.

THEOREM 2. *For a general union-preserving transformation* T, *the four variables* $(p_{n-1}, q_{n-1}, p_n, q_n)$ *can be eliminated from the three functions defining* (X, Y, Z), *thus obtaining a single eliminant of the form*

$$(6)\qquad \Omega(X, Y, Z, x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2}) = 0.$$

We call this the *directrix equation* of our general union-preserving transformation T.

Let us proceed with the proof of the above result. In the first place, it is seen by (3) that p_n and q_n can be eliminated from the first and second, and also from the first and third of our equations (1) defining the general union-preserving transformation T. We therefore obtain the two relations

$$(7)\qquad Y = F(X, x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),$$
$$Z = G(X, x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}).$$

At this point, we wish to introduce the following notation. Let X be any function of $(X, Y, Z, x, y, z, p_1, q_1, \cdots, p_m, q_m)$. Then the operation *prime* on X denotes the following linear differential operator

$$(8)\qquad \mathrm{X}' = \mathrm{X}_x + p_1\mathrm{X}_y + q_1\mathrm{X}_z + \cdots + p_{m+1}\mathrm{X}_{p_m} + q_{m+1}\mathrm{X}_{q_m}.$$

That is, X′ denotes the total derivative of X with respect to x while considering (X, Y, Z) as constants.

Next upon applying the conditions (3) for a union-preserving transformation, we discover that

$$(9)\qquad P = F_X, \qquad Q = G_X, \qquad F' = 0, \qquad G' = 0.$$