MODULARITY IN BIRKHOFF LATTICES

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The purpose of this note is to identify upper semi-modular lattices originally defined by G. Birkhoff and subsequently studied by Dilworth with those $M$-symmetric lattices (introduced independently by the author without assumption of chain conditions) which satisfy a condition of finite dimensionality.

The definitions and notations are these. In a lattice $L$, $a > b (b < a)$ means that $a$ "covers" $b$, that is, $a > b$, together with $a \geq x \geq b$ implies $x = a$ or $x = b$; $(b, c)M$ means $(a + b)c = a + bc$ for every $a \leq c$ (where $a + b, ab$ are the "join" and "meet" respectively of $a, b$). We say that $L$ is $M$-symmetric if the binary relation $M$ is symmetric; $L$ is a Birkhoff lattice if

\[ a, b > ab \text{ implies } a + b > a, b; \]

$L$ is of finite-dimensional type if for every $a < b$ there exists a finite "principal chain"

\[ a_1 < a_2 < \cdots < a_n, \]

with $a_1 = a$, $a_n = b$. When $a, b$ satisfy this condition for a specific $n$, we say that $b$ is $n - 1$ steps over $a$.

The properties of the relation $M$ are given in part in a previous paper. Additional properties needed here are contained in the following lemma.

**Lemma 1.** Suppose $b, c \in L$. Then

(a) $(b, c)M$ if and only if $bc \leq a \leq c$ implies $(a + b)c = a$;

(b) if $(b, c)M$, then $(b', c')M \text{ for } bc \leq b' \leq b, bc \leq c' \leq c$.

**Proof.** The forward implication in (a) is obvious. To prove the
converse, let \( a \leq c \). Then \( a' = a + bc \) has the property \( bc \leq a' \leq c \), whence
\[
(a + b)c = (a + bc + b)c = (a' + b)c
= a' = a + bc.
\]
To prove (b) we use the condition in (a). Let \( b'c' \leq a \leq c' \). Then
\[
(a + b')c' \leq (a + b)c' = ac' = a \leq (a + b')c',
\]
whence (b) follows.

**Theorem 1.** Every \( M \)-symmetric lattice is a Birkhoff lattice.

**Proof.** Suppose \( a, b \geq ab \). Then it is immediate that \( a + b > a, b \).
To prove \( a + b > a \), let \( a \leq c \leq a + b \). Since \( b \geq cb \geq ab \), we have \( cb = b \) or \( cb = ab \) from the hypothesis \( b > ab \). If \( cb = b \), then \( a + b \leq c \), whence \( c = a + b \). Suppose \( cb = ab \). We shall prove \( (c, b)M \). Let \( ab = cb \leq x \leq b \).
Then \( x = ab \) or \( x = b \), whence either
\[
(x + c)b = (ab + c)b = (cb + c)b = cb = x,
\]
or
\[
(x + c)b = (b + c)b = b = x,
\]
and it follows by Lemma 1 (a) that \( (c, b)M \). Now the symmetry of \( M \) yields \( (b, c)M \), and thus, since \( bc \leq a \leq c \),
\[
c = (a + b)c = a.
\]
In all cases \( c = a + b \) or \( c = a \), and consequently \( a + b > a \). Similarly \( a + b > b \).

**Remark.** The theorem just proved generalizes the known result\(^6\) that every modular lattice is a Birkhoff lattice, since modular lattices are \( M \)-symmetric.

In order to consider the converse of Theorem 1, let, for the purposes of the following lemmas, \( L \) be a fixed Birkhoff lattice of finite-dimensional type.

**Lemma 2.** If \( b, c \in L \) and \( c > bc \), then \( b + c > b \).

**Proof.**\(^7\) Observe that \( b \geq bc \); if \( b = bc \), \( b \leq c \), and \( b + c = c \geq bc = b \). If \( b > bc \), then there exists \( n = 1, 2, \cdots \) such that \( b \) is \( n \) steps over \( bc \). If \( n = 1 \), the result is obvious from condition (1) defining a Birkhoff lattice. Suppose the result has been proved for all \( b, c \) for which \( b \) is

\(^6\) Birkhoff, loc. cit. p. 34.

\(^7\) This is MacLane's second "exchange axiom" in the convex lattice of all \( x \geq bc \); as such it follows for finite-dimensional lattices from remarks on p. 63 of Birkhoff. Since \( L \) need not be finite-dimensional, we give the proof in full.
$k$ steps over $bc$, and let $b$ be $k+1$ steps over $bc$. Clearly there exists $b' < b$ such that $b'$ is $k$ steps over $bc$. Since $b'c \leq bc \leq b'c$, we have $c > b'c$, and by the induction hypothesis applied to $b'$, $c$ it follows that $b' + c > b'$. But $b' \leq (b' + c)b \leq b$, whence $(b' + c)b = b'$ or $(b' + c)b = b$. In the latter case $b' < b \leq b' + c$, and thus $b = b' + c$, whence $c \leq b$, contrary to $c > bc$. Consequently $(b' + c)b = b'$. Since $b' + c$, $b > (b' + c)b$, (1) yields

$$b + c = b + (b' + c) > b.$$  

**Lemma 3.** If $b, c \in L$, $c > bc$, then $(c, b)M, (b, c)M$.

**Proof.** If $bc \leq a \leq c$, then $a = bc$ or $a = c$, so that either

$$(a + b)c = (bc + b)c = bc = a,$$

or

$$(a + b)c = (c + b)c = c = a,$$

and $(b, c)M$. Now suppose $bc \leq a \leq b$. Then $bc \leq ac \leq bc$ yields $ac = bc$. Hence $c > ac$, and $a + c > a$ by Lemma 2. But $a \leq (a + c)b \leq a + c$, whence $(a + c)b = a$ or $(a + c)b = a + c$. In the latter case $a + c \leq b$, and $c \leq b$, which is impossible. Hence $(a + c)b = a$, and $(c, b)M$.

**Lemma 4.** Suppose $b, c \in L$, $(b, c)M$. Then $bc \leq a \leq b, a + c = b + c$ implies $a = b$.

**Proof.** If $c = bc$, that is, $c \leq b$, or if $c$ is one step over $bc$ then $(c, b)M$ either by direct verification or by Lemma 3; hence

$$a = a + cb = (a + c)b = (b + c)b = b.$$

Suppose the result holds for all $b, c$ with $n$ steps over $bc$, and let $b, c$ satisfy the hypotheses, $c$ being $n+1$ steps over $bc$. Then there exists $c'$ with $bc \leq c' < c$, where $c'$ is $n$ steps over $bc$. Since $(b, c')M$ by Lemma 1 (b), and since $bc' = bc \leq a \leq b$, we need only verify $a + c' = b + c'$ in order to show $a = b$. Since $(a, c)M$ by Lemma 1 (b), $(c' + a)c = c'$, Thus

$$c > c' = (c' + a)c,$$

and by Lemma 2,

$$c + b = c + a = c + (c' + a) > c' + a.$$

But

$$c' + a \leq c' + b \leq c + b,$$

whence

$$c' + b = c' + a \quad \text{or} \quad c' + b = c + b.$$

In the second case, since $(b, c)M$,
\[ c' = (c' + b)c = (c + b)c = c, \]

which is impossible. This completes the proof.

**Theorem 2.** Every Birkhoff lattice \( L \) of finite-dimensional type is \( M \)-symmetric.

**Proof.** Suppose \( (b, c)M \), and in proof of \( (c, b)M \) let \( bc \leq a \leq b \). Define
\[ b_1 = (a + c)b \geq a; \]
we shall prove that \( b_1 = a \) by applying Lemma 4 to \( a, b_1, c \) in place of \( a, b, c \). First, \( (b_1, c)M \) by Lemma 1 (b), since \( bc \leq b_1 \leq b \), and \( (b, c)M \). Moreover,
\[ b_1c = (a + c)cb = bc \leq a \leq b_1. \]

Finally, \( a + c \geq b_1, c \), whence
\[ a + c \geq b_1 + c \geq a + c, \]
and \( a + c = b_1 + c \). The hypotheses of Lemma 4 have been verified, and thus \( a = b_1 \), as was to be proved.

The effect of Theorems 1 and 2 is to show that not necessarily finite-dimensional \( M \)-symmetric lattices are a true generalization of the Birkhoff lattices. Moreover, the condition defining \( M \)-symmetry does not lose its strength in infinite-dimensional cases as does condition (1). For example, an interval of real numbers ordered as usual satisfies (1) vacuously; it is modular, hence \( M \)-symmetric. However, define a lattice \( L \) as consisting of the closed real interval \( I = [0, 1] \), ordered naturally, together with an element \( \epsilon \), with \( 0 < \epsilon < 1 \), but \( x < \epsilon, \epsilon < x, \epsilon \neq x \) for \( x \subseteq I \). This is a lattice in which the only covering relations are \( \epsilon > 0, 1 > \epsilon \). Hence (1) is vacuously true, but \( M \)-symmetry fails violently, since \( (x, \epsilon)M \) for every \( x \subseteq L \), but \( (\epsilon, x)M \) is false except for \( x = 0, 1 \) or \( \epsilon \).

Interesting questions are these. What infinite-dimensional generalization of the Jordan chain condition holds in \( M \)-symmetric lattices? Moreover, in finite-dimensional lattices, (1) together with its dual implies modularity; what can be said generally of lattices which together with their duals are \( M \)-symmetric?

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