BOOK REVIEW

Metric methods in Finsler spaces and in the foundations of geometry. By Herbert Busemann. Princeton University Press, 1942. 2+243 pp., 22 fig. $3.00.

Metric notions have entered the foundations of geometry in various ways. In his *Grundlagen*, Hilbert formulates axioms of congruence which imply the availability of a metric, but the idea of distance is not explicitly used. In the projective framework for the classical noneuclidean geometries (for example, Klein), distance is defined in terms of specific configurations which are themselves the product of rather elaborate preparatory study. In the differential approach (for example, Cartan), the theory of coördinate manifolds is taken for granted, distance is first introduced locally, and even after its extension to nonlocal measurements it plays little essential part in the investigations. Others (Lie, Hilbert (Anhang IV to the *Grundlagen*)) based their work on a group of transformations (motions) which provided the means for defining congruence. The advent of metric spaces (Fréchet) raised the possibility of beginning with metric ideas and developing from them the equipment and results of the earlier theories. Menger isolated and studied the properties a metric space must have in order to possess geodesics which behave conveniently. The present monograph takes up at that point and develops a connected theory, occasional portions of which have appeared elsewhere, of metric spaces with geodesics.

The object of study is a finitely compact (hence separable) metric space Σ which is (internally) convex in the sense that, if $X \neq Z$, there is a point Y between X and Z (that is, such that $XY+YZ=XZ$). Define a segment to be a congruent map of a closed real interval, and a geodesic to be a locally congruent map of the real axis. As a decisive local restriction (D), each point of Σ is assumed to have a neighborhood N such that, if A and B are any distinct points of N, the set of points between A and B forms a segment which can be extended appreciably (in fact, outside N), in either direction past A and B, and which is the *unique* segment between any pair of its points of which neither is too far beyond A or B. Without Axiom D, any two points P and Q can be joined by a segment. With Axiom D, this segment turns out to be unique, if P and Q are not too far apart; further, any pair can be connected by a geodesic, and a given segment can be embedded in one and only one geodesic. If there is a point at which

Σ is one-dimensional, Σ is congruent to a euclidean straight line or circle; if Σ is two-dimensional, Σ is a manifold.

Σ is Minkowskian at a point P if and only if there is a neighborhood N of P into which a metric can be introduced, equivalent in an appropriate sense to the original metric, in such a way that under the new metric N is congruent to an open sphere of a linear space which is so normed (cf. Minkowski) that its unit sphere U is a strictly convex surface with the origin as center. The author outlines a proof that a Finsler space Φ is a space Σ which is Minkowskian at each point if Φ is regular (because U is strictly convex) and symmetric (because U has a center). Starting off, conversely, with a space Σ (of chap. 1), he imposes a local condition (Δ) which enables him to introduce a Minkowski metric m at each point where Δ is satisfied. Δ refers to a trio of points which tend to coincidence, and controls the convergence of the segments and of the ratios of the distances determined by these points. The metric m is defined as follows, in a suitably restricted neighborhood N of P: For A in N and $0 \leqq t \leqq 1$, let A_t be that point on the segment $\overline{PA}$ for which $PA_t = tPA$; then $m(A, B) = \lim t^{-1}A_tB_t$ as $t \to 0+$. The neighborhood N is expanded to a space Σ_P by continuing each segment $\overline{PA}$ past A up to the boundary of N and then on indefinitely through new points created explicitly for this purpose. $m(A, B)$ is extended throughout Σ_P, which is then seen to be a normed (Minkowskian) linear space associated with Σ at P. It follows from a further local condition Δ', if an open set G of Σ is the intersection of Σ_P and Σ_Q, that the local (normal) coordinates provided for G by these spaces are connected by a transformation of class 1. Finally, the metric m is used to define a function $F(x, \lambda)$ of the sort with which Finsler theories commonly set out. Thus Δ and Δ' are conditions sufficient to insure that Σ be a regular symmetric Finsler space. It may happen that m is euclidean in Σ_P; in the important special case in which this happens at every point, the Finsler space reduces to a Riemann space and Σ_P then differs but slightly from the tangent space widely used in differential geometry.

Specializing the space Σ of chap. 1 in quite a different manner, the remainder (two-thirds) of the book is based on the global Axiom E: Any two distinct points are on at most (hence exactly) one geodesic. Each geodesic of such a (straight line, S.L.) space is either open or closed (congruent to a euclidean straight line or circle), and the closed (open) geodesics form an open (closed) set. In the two-dimensional case any two closed geodesics intersect, and the geodesics through a given point are either all closed or all open. Hence an S.L. plane is either closed (all geodesics closed) or open (all geodesics open). By mapping an S.L. plane Σ in a specific way on a projective plane P^2 with an elliptic metric, it is shown that Σ is homeomorphic to E^2 (euclidean plane) or P^2 according as Σ is open or closed. A homeomorphism carrying geodesics of Σ into straight lines is possible (that is, Σ is Arguesian) if and only if Desargues' Theorem holds. (An Arguesian S.L. space of any dimension is either closed or open.) The inverse problem for E^2 seeks conditions on a family F of curves α in E^2 sufficient to insure that a topologically equivalent new metric can be introduced in E^2 in such a way that the curves α become the geodesics of the S.L. plane thus defined. In a very satisfying way, the sufficiency of the following remarkably weak conditions is established: each α is an open simple unbounded Jordan curve, and any two distinct points of E^2 are contained in eactly one curve of F.

Define a sphere to be convex if no geodesic tangent to it contains a point interior to it. Let each sphere of the S.L. space Σ be convex (condition K). This implies, for fixed P, as X moves on a geodesic g, that PX either is constant or has precisely one minimum which is attained just once. It follows that Σ is either open or closed. In the closed case, with dimension not less than 3, Σ is elliptic. In the open case K is equivalent to convexity in the usual sense and to the existence of precisely one (suitably defined) perpendicular to each geodesic from each external point. By detailed argument in which perpendiculars and baselines, parallels, bisectors (loci $PX=QX$) and their limits, and so on, are studied extensively, several theorems are proved establishing conditions which completely characterize Σ in one way or another. Examples: with dimension not less than 3, K, differentiable spheres (suitably defined), and euclidean parallelism, Σ is Minkowskian. In the plane case, with euclidean parallelism, Σ is Minkowskian if the following strengthened form of K holds: for some fixed $\alpha \geqq 1$, if $XY=YZ=XZ/2$, then $2PY^\alpha \leqq PX^\alpha+PZ^\alpha$ for any P. If all bisectors are linear, regardless of dimension, then Σ is euclidean or hyperbolic.

Along with the classical geometries, n-dimensional S.L. spaces Σ can be characterized in terms of their mobility. An examination of involutoric motions and their fixed points reveals that Σ is homogeneous (congruent to a euclidean, hyperbolic, or elliptic space) if it is symmetric about each geodesic. If Σ is closed and symmetric about each point, Σ is elliptic. In E^n a congruence between two triangles can be extended to a motion of E^n; if Σ has the property that there is a number $\delta>0$ such that a congruence between any two triangles, each with diameter less than δ, can be extended to a motion of Σ, then Σ is homogeneous. An example of Fubini is adapted to show that the similar hypothesis for pairs instead of trios of points is insufficient. Suppose that Σ is open, and admits a transitive abelian group G of motions; it is easily seen that G is closed and simply transitive and generates a Minkowskian plane by carrying either of two intersecting geodesics along the other, whence it follows that Σ is Minkowskian. The plane case is studied in detail by means of translations along geodesics. Typical result: An S.L. plane is Minkowskian, hyperbolic, or elliptic if all possible translations exist along two geodesics one of which is not an asymptote to either orientation of the other. If an S.L. plane admits a transitive group of motions, then: either the plane is closed and its metric elliptic, or parallelism is euclidean and the metric is Minkowskian, or parallelism is hyperbolic and the metric is quasi-hyperbolic (admitting all translations along a geodesic and its nonparallel asymptotes). Attention is called, in conclusion, to a number of unsolved problems.

As the cited results suggest, a superficial refresher in non-euclidean geometry is a desirable preliminary. The exposition is clear and persuasive with the exception of one passage (pp. 49–63) which the reviewer found rough in spots. The situation there is inherently complicated, notation, typography, and display are uncoöperative, and the style seems unnecessarily terse. Slight expansion would have rendered this material much more pleasantly accessible.

Reading is enlivened by a stimulating diffusion of misprints. Only the following, which the author has kindly verified, are thought capable of delaying the attentive reader:

Page	*Line*	*For*	*Read*
9	7	4	5
65	4′	A, B	A_ν, B_ν
97	4	$[tf(n)/n-1/m2^n]$	$t[2f(n)/n-1/m2^n]$
167	8	h	π

With these trivial qualifications, the project has been conceived and executed in a thoroughly workmanlike fashion. Loose ends lead either to specified passages in the literature, or to outstanding problems which are clearly indicated with accompanying remarks and references. Although the argument is basically "coördinate-free," appealing picturesquely to the intuition, one senses none of that loss of rigor which is sometimes felt to accompany "synthetic" methods. In presenting a connected account of the author's valuable contributions, this book gives very useful access to an interesting field of study.

In a recent paper (Trans. Amer. Math. Soc. vol. 54 (1943) pp. 171–184) the author establishes important new results.

1. A closed S.L. space of dimension not less than 3 is elliptic if each of its spheres has order 2 (condition K_1, weaker than K). Thus K_1, which was known (p. 135) to be equivalent to K in open spaces, is now seen to be an adequate basis for the main theorem (p. 124, cited above) proved with K in closed spaces.

2. Several conditions, of which a few have been cited, are found in the book under which an S.L. space is necessarily open or closed. Such conditions are now seen to be entirely superfluous, for any S.L. space is either open or closed with each geodesic in the closed case having the same length.

These facts open the way for substantial improvements in the study based in the book on condition K.

F. A. FICKEN