CLUSTER POINTS OF SUBSEQUENCES

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In the preceding paper [1] Buck defines a class of "subsequences" of a multiple sequence and shows that "almost all" of such subsequences have certain properties. This note is essentially based on a different choice of the definition of "subsequences"; that is, this paper and [1] are generalizations in different directions of a preceding paper by Buck and Pollard (reference 2 of [1]). In this discussion countability is the important property of the index systems such as the integers underlying the simple sequences or the \( n \)-tuples of integers underlying the multiple sequences. Countability is a slightly stronger condition than is necessary since the results will be shown to hold as well for functions of \( n \) variables as for multiple sequences; some other special cases are mentioned at the end of this paper. Also I modify Buck's approach by considering cluster points in neighborhood spaces rather than limit points in convergence spaces [3]. It may be mentioned that even for multiple sequences Theorems 1 and 2 of these papers are independent since Buck's set of "subsequences" is a set of measure zero in the set of "subsequences" considered here; my Theorem 3 contains the corresponding theorem of [1] as a special case. Lemma 1 and its corollary, Lemma 3, are the fundamental results on which the theorems rest; Lemma 3 is the generalization appropriate to this paper of the lemma in §3 of [1].

1. Preliminaries. If \( R \) is any set, a product measure can be defined in the set of characteristic functions of subsets of \( R \) [1, footnote 2] and this in turn induces a measure \( | \cdot | \) for subsets of the set \( \mathcal{E} \) of all subsets \( E \) of \( R \); this measure is non-negative, completely additive, and (if \( R \) is infinite) takes all values between 0 and 1 inclusive; its other principal characteristic is that if \( r_1, \ldots, r_k \in R \), then \( \{ E \mid \text{no } r_i \in E \} \) is of measure \( 2^{-k} \); hence if \( E_0 \) is an infinite subset of \( R \) and \( A = \{ E \mid E \cap E_0 \text{ is empty} \} \), \( | A | = 0 \).

An index system \( \mathcal{R} = (R, \geq) \) is a set \( R \) and a binary relation \( \geq \) such that \( \geq \) is transitive and every element \( r_0 \) has a successor \( r_1 > r_0 \) such that \( r_0 \geq r_1 \). (In the language of [4] \( \mathcal{R} \) is oriented and has no terminal

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1 Considerations suggested by the preceding paper of R. C. Buck.
2 Numbers in brackets refer to the Bibliography at the end of the paper.
3 The usual notation of \( \bigcup \) and \( \bigcap \) will be used for union and intersection of sets; \( \{ p \mid P \} \) will mean the set of all \( p \) having the property \( P \).
elements.) A set $E$ in $\mathcal{R}$ is called \textit{cofinal} in $\mathcal{R}$ if for every $r$ in $\mathcal{R}$ there exists $r' \geq r$ with $r' \in E$. Let $E^* = \{r | r \geq r' \text{ for some } r' \in E\}$.

Note that if $\mathcal{R}$ is the system of integers ordered by magnitude, then the cofinal subsets of $\mathcal{R}$ are the infinite subsets; for $g$ defined on a general index system $\mathcal{R}$ it is clear that reducing the domain of definition of $g$ to a cofinal subset $E$ of $\mathcal{R}$ is a generalization of the process of selecting a subsequence in case $\mathcal{R}$ is the system of integers.

A subsystem $\mathcal{R}' = (R', \geq)$ of $\mathcal{R} = (R, \geq)$ is a subset $R'$ of $R$ with the order relation between points of $R'$ defined by that in $R$; if $R'$ is cofinal in the index system $\mathcal{R}$, then $(R', \geq)$ is also an index system. Cofinality is transitive in a transitive system; that is, if $R'$ is cofinal in $(R, \geq)$ and $R''$ is cofinal in $(R', \geq)$, then $R''$ is cofinal in $(R, \geq)$.

We may note that if $\mathcal{R}$ is the set of $n$-tuples of integers (the case studied in [1]), where $(i_1, \ldots, i_n) \geq (j_1, \ldots, j_n)$ if and only if $i_k \geq j_k$ for every $k \leq n$, then the product subsets defined by Buck are cofinal in $\mathcal{R}$ but are very sparsely distributed in the set of all cofinal subsets of $\mathcal{R}$; to be precise, such sets form a set of measure 0 if $n \geq 2$. A product set in the set of $n$-tuples of integers, $\mathcal{R} = I \times I \times \cdots \times I$, is a set of the form $E_1 \times E_2 \times \cdots \times E_n$, $E_i \subseteq I$; the product sets define the class of "subsequences" used by Buck. If $E_k$ is the set of elements of $\mathcal{R}$ with all coordinates not greater than $k$ and if $A_k$ is the class of all subsets $E$ of $\mathcal{R}$ such that $E \cap E_k$ is a product set in $E_k$, then $A = \bigcap_k A_k$. It is easily seen that if $E' \subseteq E_k \{E | E \cap E_k = E'\}$ is of measure $2^{-n_k}$; since there are $(2^k - 1)^n + 1 < 2^{n_k}$ product sets in $E_k$, it follows that $|A_k| < 2^{n_k - k^2}$; if $n \geq 2$, this tends to zero as $k$ increases so $|A| = 0$.

**Lemma 1.** If $\mathcal{R}$ has a countable cofinal subset and $\mathcal{C}$ is the set of all cofinal subsets of $\mathcal{R}$, then $|\mathcal{C}| = 1$.

Let $\mathcal{R}'$ be a countable cofinal subset of $\mathcal{R}$ and suppose $E \in \mathcal{C}$; then there exists $r$ in $\mathcal{R}$ such that $(r)^* \cap E$ is empty. Since there exists $r'$ in $\mathcal{R}'$ such that $r' \geq r$, it follows that $(r')^* \cap E$ is empty. Since $r'$ has an infinite number of distinct successors in $\mathcal{R}'$, the set $A_{r'} = \{E | (r')^* \cap E$ is empty} is of measure zero. Since $\mathcal{E} - \mathcal{C} = \bigcup_{r' \in \mathcal{R}'} A_{r'}$, $|\mathcal{E} - \mathcal{C}| = 0$ so $|\mathcal{C}| = 1$.

By means of this lemma we can define a measure in $\mathcal{C}$ by taking the measure in $\mathcal{E}$ of elements of $\mathcal{C}$; since $|\mathcal{C}| = 1$, we can talk meaningfully about almost all cofinal subsets of $\mathcal{R}$ [5, Theorem 1.1]. Note that cofinality of $E$ is not affected by adding or removing a finite set, so $\mathcal{C}$ is a "homogeneous" subset of $\mathcal{E}$ and therefore if it is measurable must have measure 0 or 1; which case occurs when $\mathcal{R}$ does not have a countable cofinal subset, I do not know.

$X$ is a \textit{neighborhood space} [3] if for each $x$ in $X$ is defined a non-
empty family of subsets of $X$, the neighborhoods of $x$. If $g$ is a function defined on an index system $\mathcal{R}$ with values in a neighborhood space $X$, $x$ is a limit point of $g$ (symbol: $x = \lim_{\mathcal{R} \to x} g$) if for each neighborhood $N$ of $x$ and every $r_0$ in $R$ there exists $r_1 \geq r_0$ such that $g(r) \in N$ whenever $r \geq r_1$. (This definition is due to Alaoglu and Birkhoff [2]; in case $\mathcal{R}$ is directed it reduces to the standard simpler form: $x = \lim_{\mathcal{R} \to x} g$ if for each $N$ there exists $r_N$ in $R$ such that $g(r) \in N$ whenever $r \geq r_N$. $\mathcal{R}$ is directed if every pair of elements has a common successor.) A point $x$ is called a cluster point of $g$ if for every neighborhood $N$ of $x$ and every $r_0$ in $R$ there exists $r_1 \geq r_0$ such that $g(r_1) \in N$. Clearly every limit point of $g$ is a cluster point of $g$, but not conversely. (See Lemma 2 below.) If $g$ is a function from $\mathcal{R}$ into $X$ and $\mathcal{E}$ is a cofinal subset of $\mathcal{R}$, let $g_\mathcal{E}$ be the function $g$ reduced onto $\mathcal{E}$ and let $Qg_\mathcal{E}$ be the set of cluster points of $g_\mathcal{E}$; the function $g_\mathcal{E}$ and the set $Qg_\mathcal{E}$ will play a role here analogous to that played by the subsequence $x'$ and the set $Px'$ in [1]. Clearly $x = \lim_{\mathcal{R} \to x} g$ implies $x = \lim_{\mathcal{E} \to x} g_\mathcal{E}$ for every $\mathcal{E}$ cofinal in $\mathcal{R}$. Let $Pg_\mathcal{E}$ be the set of limit points of functions $g_{\mathcal{E}'}$ for $\mathcal{E}'$ cofinal in $\mathcal{E}$; that is, $x \in Pg_\mathcal{E}$ if and only if there exists $\mathcal{E}'$ cofinal in $(\mathcal{E}, \succeq)$ such that $x = \lim_{\mathcal{E}' \to x} g_{\mathcal{E}'}$.

Recall that $X$ is said to satisfy Hausdorff's first countability condition if for each $x$ in $X$ there is a countable set $\{N_i\}$ of neighborhoods of $x$ such that each neighborhood of $x$ contains an $N_i$. The next lemma shows the connection between $Qg$ and $Pg$.

**Lemma 2.** If $\mathcal{R}$ has a countable cofinal subsystem, if $X$ satisfies the first countability condition, and if the intersection of each pair of neighborhoods of each point $x$ of $X$ contains a third neighborhood of $x$, then $x$ is a cluster point of $g$ if and only if there exists $\mathcal{E}$ cofinal in $\mathcal{R}$ such that $x = \lim_{\mathcal{E} \to x} g_\mathcal{E}$; that is, $Qg = Pg$.

$Qg \supset Pg$ with no restriction on $R$ or $X$, for $x = \lim_{\mathcal{E} \to x} g_\mathcal{E}$ and $N$ a neighborhood of $x$ imply that if $r \in R$, there exists $r_1 \in E$ with $r_1 \geq r$ and then an $r_2$ in $E$ such that $r_2 \geq r_1$ and $g(r_2) \in N$. If $\mathcal{R}$ and $X$ are restricted as above and if $x$ is a cluster point of $g$, there exists a sequence $\{N_i\}$ of neighborhoods of $x$ such that each neighborhood of $x$ contains an $N_i$. By the other condition there exists a decreasing sequence of such neighborhoods $N_1 \supset N_2 \supset \cdots \supset N_i \supset \cdots$. Enumerate $R$ in a sequence $\{r_j\}$; then let $r_1$ be a point of $g^{-1}(N_1)$ which follows $r'_1$; let $r_2$ and $r_3$ be points of $g^{-1}(N_2)$ which, respectively, follow $r_1$ and $r'_1$; let $r_4$, $r_5$, $r_6$, $r_7$ be points of $g^{-1}(N_3)$ which follow $r_2$, $r_3$, and $r'_2$, and so on. Then $E = \{r_i\}$ contains a successor of every element of $R'$, so is cofinal in $(\mathcal{R}', \succeq)$ and hence cofinal in $\mathcal{R}$. If $N$ is a neighborhood of $x$, there is an $N_i \subset N$ and there exists $n$ such that $g(r_i) \in N_j$ if $i \geq n$. 
Since the set of all r which do not precede any \( r_i, i < n \), is cofinal in \( R \) and contains all successors of each of its elements, its intersection with \( E \) is a set of the same sort in \( E \); this shows that \( E \) has the desired property; that is, that \( x = \lim_{(E, \succeq)} \{ g \} \).

Note that no such relation holds for multiple sequences if the cofinal sets of \( R \) which are used are restricted as in \([1]\) to be product sets.

We used Lemma 1 to show that "almost everywhere" has meaning in \( C \); a simple application of the same proof gives the next result which can be regarded as an extension of the lemma of \([1, \S 3]\].

**Lemma 3.** If \( R \) has a countable cofinal subsystem, if \( E_0 \) is cofinal in \( R \), and if \( A = \{ E \mid E \cap E_0 \) is not cofinal in \( E_0 \} \), then \( |A| = 0 \); that is, almost every \( E \) of \( C \) meets \( E_0 \) in a set cofinal in \( R \).

Let \( E_1 \) be a countable subset of \( E_0 \) cofinal in \((E_0, \succeq)\); then \( E \cap E_0 \) not cofinal in \( E_0 \) means that there exists \( r_E \in E_1 \) such that \( E \cap E_0 \cap (r_E)^* \) is empty. For fixed \( r \in E_1 \) let \( A_r = \{ E \mid E \cap E_0 \cap (r)^* \) is empty \} ; since \( E_0 \cap (r)^* \) is infinite, \( |A_r| = 0 \); since \( A = \bigcup_{r \in E_1} A_r \), \( |A| = 0 \).

2. **Cluster points.** We now proceed to the analogues of the theorems of \([1]\).

**Theorem 1.** If \( R \) is an index system with a countable cofinal subset, if \( X \) satisfies the first countability condition, if \( g \) is a function from \( R \) into \( X \), and if \( x \in Qg \), then \( x \in Qg_E \) for almost every \( E \) of \( C \); that is, each cluster point of \( g \) is a cluster point of almost every \( g_E \).

\( x \) is a cluster point of \( g \) if and only if \( g^{-1}(N) \) is cofinal in \( R \) for every neighborhood \( N \) of \( x \). If \( \{ N_i \} \) is an equivalent sequence of neighborhoods of \( x \), let \( A_i = \{ E \mid E \cap g^{-1}(N_i) \) is not cofinal in \( g^{-1}(N_i) \} \); then, by Lemma 3, \( |A_i| = 0 \). Setting \( A = C - \bigcup_i A_i \), \( |A| = |C| = 1 \).

If \( E \subseteq A \) and \( N \) is a neighborhood of \( x \), there is an \( N_i \subseteq N \); since \( E \cap A_i \), \( E \cap g^{-1}(N_i) \) is cofinal in \( g^{-1}(N_i) \) and hence cofinal in \( R \). Since \( E \cap g^{-1}(N) \supseteq E \cap g^{-1}(N_i) \), \( E \cap g^{-1}(N) \) is also cofinal in \( R \) and therefore is cofinal in \((E, \succeq)\); that is, if \( N \) is a neighborhood of \( x \) and \( E \subseteq A \), \( g^{-1}(N) \cap E \) is cofinal in \((E, \succeq)\), that is, \( x \) is a cluster point of \( g_E \) if \( E \subseteq A \).

Limit points have an analogous property.

**Theorem 1'.** If \( R \) and \( X \) satisfy the hypotheses of Theorem 1, then \( x = \lim_{(R, \succeq)} \{ g \} \) if and only if \( x = \lim_{(E, \succeq)} g_E \) for almost every \( E \) in \( C \).

If \( x = \lim_{(R, \succeq)} \{ g \} \), then \( x = \lim_{(E, \succeq)} g_E \) for every \( E \) in \( C \). If \( x \neq \lim_{(R, \succeq)} \{ g \} \),
there exists a neighborhood $N$ of $x$ and an $r_0$ in $\mathbb{R}$ such that every $r_1 > r_0$ has a successor $r_2 > r_1$ for which $g(r_2) \in N$; let $E_0 = \{ r | g(r) \in X - N \text{ and } r > r_0 \}$; then if $E_1$ is so chosen that $E_0$ and $E_1$ have no common successors and $E_0 \cup E_1$ is cofinal in $\mathbb{R}$, by Lemma 3 the set $A = \{ E | E \cap (E_0 \cup E_1) \text{ is cofinal in } \mathbb{R} \}$ is of measure 1. For any such $E$, $E \cap E_0$ is cofinal in $(E_0, \supseteq)$ so $x \neq \lim_{(E, \supseteq)} g_E$ if $E \subseteq A$; that is, $x \neq \lim_{(E, \supseteq)} g$ implies $x \neq \lim_{(E, \supseteq)} g_E$ for almost every $E$ in $\mathcal{C}$.

Say that $g$ is divergent if $x = \lim_{(E, \supseteq)} g$ is false for every $x$ in $X$.

**COROLLARY.** Let $X$ and $\mathbb{R}$ satisfy the conditions of the theorem and suppose that $g$ is divergent; then for each $x$ in $X$ the set $A_x = \{ E | x = \lim_{(E, \supseteq)} g_E \}$ is of measure zero. Hence if almost every $g_E$ has a limit point, then $P_g$ is uncountable.

The first statement follows immediately from the theorem. For the second, $\{ E \mid g_E \text{ has a limit point} \} = \bigcup_{x \subseteq P_x} A_x$; since $|A_x| = 0$ and $\bigcup_{x \subseteq P_x} A_x = 1$, $P_g$ is uncountable.

The next two results are related to Theorem 1' but stronger hypotheses enable us to draw stronger conclusions.

**Theorem 2.** If $X$ and $\mathbb{R}$ satisfy the conditions of Theorem 1 and if each pair of distinct points of $X$ has a pair of disjoint neighborhoods, then $g$ is divergent if and only if almost every $g_E$ is divergent.

If $g$ has the limit $x$, so does every $g_E$. If $g$ is divergent, by the corollary $|A_x| = 0$ for every $x$. By the first statement in the proof of Lemma 2, if $x_1 = \lim_{(E, \supseteq)} g_E$, then $x_1$ is a cluster point of $g$; by Theorem 1, $x_1$ is a cluster point of almost every $g_E$. Let $A = \{ E \mid x_1 \neq \lim_{(E, \supseteq)} g_E \text{ but } g_E \text{ has a limit point} \}$. If $E$ is in $A$, let $x = \lim_{(E, \supseteq)} g_E$; since there exist disjoint neighborhoods $N_1$ of $x_1$ and $N$ of $x$ and since $g_E$ plunges eventually into $N$, there is an $r_1$ in $E$ such that $g_E(r) \in N_1$ if $r > r_1$ and $r \in E$. Hence $g_E^{-1}(N_1)$ is not cofinal in $(E, \supseteq)$, so $x_1$ is not a cluster point of $g_E$ when $E \subseteq A$. Hence $|A| = 0$ by Theorem 1; since $|A_x| = 0$ also, we see that $|\{ E \mid g_E \text{ has a limit} \}| = |A| + |A_x| = 0$.

Buck notes that the proof of Theorem 1' can easily be modified to prove another theorem with the same conclusion as that of Theorem 2.

**Theorem 2'.** If $\mathbb{R}$ has a countable cofinal subset and if $X$ satisfies Hausdorff's second countability condition, then $g$ is divergent if and only if almost every $g_E$ is divergent.

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4 This is the separation condition in a Hausdorff space; however $X$ need not satisfy the other axioms of such a space.

5 In this system the second countability condition becomes: There exists a countable subset $\{ N_i \}$ of subsets of $X$ such that for each $x$ and each neighborhood $N$ of $x$ there is an $i$ such that $N_i$ contains a neighborhood of $x$ and $N_i \subseteq N$. 

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Lemma 4. If every neighborhood \( N \) of \( x \) contains a neighborhood \( N' \) of \( x \) such that for every \( y \) in \( N' \) there is a neighborhood \( N_y \) of \( y \) with \( N_y \subset N \), then \( Qg \) is closed in \( X \).

If \( x \) is in the closure of \( Qg \), then for every neighborhood \( N \) of \( x \) there is a point \( y \) in \( N' \cap Qg \); then for every \( r_0 \) in \( R \) there exists \( r_1 \geq r_0 \) such that \( g(r_1) \subset N_y \subset N \) so \( x \in Qg \).

Theorem 3. If \( \mathcal{R} \) and \( X \) satisfy the hypotheses of Theorem 1 and Lemma 4, and if \( Qg \) is separable, then \( Qg = Qg_E \) for almost every \( E \) in \( \mathcal{C} \).

Take a countable dense subset \( X' \) of \( Qg \) and follow the proof of Theorem 3 of [1], using Theorem 1 and Lemma 4 at the appropriate points. This is much stronger than the corresponding theorem of [1]; the principal extension is that this formulation is valid for all essentially countable index systems rather than for the integers alone. In case \( \mathcal{R} \) is the system of integers, this result includes that of [1] since the hypotheses of [1, Theorem 3] imply the hypotheses of Theorem 1 and Lemmas 2 and 4; Theorems 1 and 2 are not generalizations of the corresponding results of [1] but rather are generalizations in a slightly different direction from the case \( n = 1 \) of those theorems.

Note that a metric space satisfies all the hypotheses on \( X \) except that on \( Qg \) in Theorem 3; there the requirement that \( X \) is separable would be a sufficient additional condition. Hence with \( X \) metric and \( \mathcal{R} \) having a countable cofinal subset, the set \( Qg \) used in this paper is equal to the set \( P_g \) analogous to \( P_x \) of [1]. Any countable index system will do for \( \mathcal{R} \) as will the system of real numbers ordered by magnitude or the system of \( n \)-tuples of real numbers ordered by magnitude or the system of \( n \)-tuples of real numbers ordered by \( (a_1, \ldots, a_n) \geq (b_1, \ldots, b_n) \) if \( a_i \geq b_i \) for all \( i \leq n \). Another such system is the system of \( n \)-tuples \( (r_1, \ldots, r_n) \) where \( r_i \in \mathcal{R}_i \), an index system with a countable cofinal subset, and where \( (r_1, \ldots, r_n) > (r'_1, \ldots, r'_n) \) if \( r_i > r'_i \) or \( r_1 = r'_1 \) and \( r_2 > r'_2 \) or, for some \( j \leq n \), \( r_i = r'_i \) for \( i < j \) while \( r_j > r'_j \). (This is the so-called ordinal or lexicographic product of the systems \( \mathcal{R}_i \).) Still another example is the system of pairs of integers where \( (i_1, i_2) \geq (j_1, j_2) \) means that \( i_1 = i_2 \) and \( j_1 \geq j_2 \).

It may be noted by means of Lemmas 1 and 3 that the proofs of Theorems 1 and 2 of [1] also hold when the set \( I \times I \times \cdots \times I \) used in [1] as the domain of the function \( x = x[i_1, i_2, \ldots, i_n] \) is replaced by \( \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n \), where the \( \mathcal{R}_i \) are any index systems with countable cofinal subsystems, providing that \( \mathcal{S} \) is then defined as \( \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n \) where \( \mathcal{C}_i \) is the family of cofinal subsets of \( \mathcal{R}_i \). The theorems thus obtained almost include the corresponding results of both papers, the case \( n = 1 \) giving analogues of the present theorems.
with stronger hypotheses, the case where all $R_i = I$ giving those of [1].

An open question is whether the existence of a countable cofinal subset is needed to derive the conclusions of Lemmas 1 and 3; if not, some weakening of the hypotheses of the theorems would be possible.

**BIBLIOGRAPHY**


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