A NOTE ON RIESZ SUMMABILITY OF THE TYPE $e^{n^\alpha}$

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Recently I proved the following result in the case $r = 2$ (Wang [4]).

Let $\sigma_n^{(r)}$ be the $r$th Cesàro mean of the series $\sum_{n=0}^{\infty}a_n$. If $\sigma_n^{(r)} - s = o(n^{-r\alpha})$ $0 < \alpha < 1$, as $n \to \infty$, where $r$ is a positive integer, and $a_n > -Kn^{\alpha-1}$, the series converges to sum $s$.

For the case $r = 1$ this result has been established by Boas [1]. His argument, however, does not seem to be applicable in any simple way to the general case.

The object of this note is to prove a theorem on Riesz summability of type $e^{n^\alpha}$, and then to deduce the result above from a Tauberian theorem of Hardy [2].

Let us put $C_r(\omega) = a_0e^{\omega^\alpha} + \sum_{n=0}^{\infty} (e^{\omega^\alpha} - e^{n^\alpha})^r a_n$. A series $\sum_{n=0}^{\infty}a_n$ is said to be summable $(e^{n^\alpha}, \tau)$ to the sum $s$ if

$$C_r(\omega) = s\rho\alpha + o(e^{\omega^\alpha}).$$

The result by Hardy which is to be called upon is the following: If the series $\sum_{n=0}^{\infty}a_n$, with terms $a_n \geq -Kn^{\alpha-1}$, $0 < \alpha < 1$, is summable $(e^{n^\alpha}, \tau)$, it is convergent. We shall now prove the following theorem.

**Theorem.** If $\sigma_n^{(r)} - s = o(n^{-r\alpha})$, $0 < \alpha < 1$, as $n \to \infty$, the series $\sum_{n=0}^{\infty}a_n$ is summable $(e^{n^\alpha}, \tau)$ to the sum $s$, where $\tau > r/(1-\alpha)$.

To prove this let $\beta_n = (e^{n^\alpha} - e^{n^\alpha})^r$, $\Delta \beta_n = \beta_n - \beta_{n+1}$, $\Delta^{r+1} \beta_n = \Delta^r \beta_{n+1}$ and

$$s_n^{(r)} = \sum_{\nu=0}^{n} \binom{n-\nu+r}{n-\nu} a_\nu,$$

$m = [\omega]$. Then, by successive Abel’s transformations we have

$$C_r(\omega) = a_0e^{\omega^\alpha} + \sum_{n=1}^{m} \beta_n a_n$$

$$= a_0e^{\omega^\alpha} + \sum_{n=1}^{m-r-1} s_n^{(r)} \Delta^{r+1} \beta_n + \sum_{i=0}^{r} s_0^{(i)} \Delta^i \beta_{m-i} - \sum_{i=0}^{r} s_i^{(i)} \Delta^i \beta_{1}$$

$$= a_0e^{\omega^\alpha} + J_1 + J_2 - J_3.$$

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1 Numbers in brackets refer to the references listed at the end of the paper.
Since \( \beta_{m-i} = (e^{\omega a} - e^{(m-i)\omega})^r = O(e^{\tau ra^\alpha} w^{(a-1)}), \) it follows that
\[
\Delta^{(i)} \beta_{m-i} = O(e^{\tau ra^\alpha} w^{(a-1)}) \quad \text{for } 0 \leq i \leq r.
\]
By a familiar theorem on Cesàro sums we get
\[
s^{(i)}_{m-i} = O(a^r), \quad \text{for } 0 \leq i \leq r,
\]
and from this
\[
J_2 = O(e^{\tau ra^\alpha} w^{(a-1)+r}) = o(e^{\tau ra^\alpha}).
\]
Since \( \Delta^i \beta = O(e^{\tau ra^\alpha} w^{(a-1)}), \) for \( 1 \leq i \leq r, \) and \( \beta_1 = (e^{\omega a} - \omega), \ s_0^{(i)} = s_0 = a_0 \) we get
\[
J_3 = e^{\tau ra^\alpha} a_0 + o(e^{\tau ra^\alpha}).
\]
By the hypothesis of the theorem we have
\[
J_1 = \sum_{n=1}^{m-r-1} s_n^{(r)} \Delta^{r+1} \beta_n = s \sum_{n=1}^{m-r-1} \left( \binom{n+r}{n} \right) \Delta^{r+1} \beta_n
\]
\[
+ o \left( \sum_{n=1}^{m-r-1} n^{r(1-\alpha)} | \Delta^{r+1} \beta_n | \right).
\]
It follows by mathematical induction that
\[
\Delta^{r+1} \beta_n = (-1)^{r+1} \int_0^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_r}^{x_r+1} B^{(r+1)}(x_{r+1}) dx_{r+1},
\]
where
\[
\beta^{(r+1)}(x) = \frac{d^{r+1}}{dx^{r+1}} \left\{ (e^{\omega a} - e^{\alpha x})^r \right\}.
\]
By direct differentiation we have
\[
B^{(r+1)}(x) = \sum_{j=1}^{r+1} \sum_{i=1}^{r} c_{ij} \Psi^{(r-j)}(x) e^{i \alpha x} x^{i-1},
\]
where \( \tau \) is taken as a positive integer and the constants \( c_{ij} \) depend upon \( i, j, \tau, r, \alpha. \) Hence we get
\[
\Delta^{r+1} \beta_n = \sum_{i=1}^{r} O(e^{(r-i)\omega a} e^{i \alpha n^{(r+1)(a-1)}}).
\]
It is easily verified by Abel's transformation that
\[
\sum_{n=1}^{m-r-1} \left( \binom{n+r}{n} \right) \Delta^{r+1} \beta_n = e^{\tau ra^\alpha} + o(e^{\tau ra^\alpha}).
\]
Hence by (5), (6), and (7)

\[ J_1 = s e^{\alpha x} + o(e^{\alpha x}) + o \left( \sum_{i=1}^{m} \sum_{n=1}^{r-1} e^{(r-i)\alpha x} e^{in\alpha x} \right) \]

\[ = s e^{\alpha x} + o(e^{\alpha x}). \]

The proof of the theorem follows from (4), (2), (3), and (8).

I conclude by observing that the theorem is the best possible of its kind (Wang [4]).

References


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