SCHUR'S THEOREMS ON COMMUTATIVE MATRICES

N. JACOBSON

In 1905 I. Schur proved that the maximum number $N(n)$ of linearly independent commutative matrices of $n$ rows and columns is given by the formula $N(n) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1 = \nu^2 + 1$ if $n = 2\nu$ and $= \nu(\nu - 1) + 1$ if $n = 2\nu - 1$. Schur also determined the sets of linearly independent commutative matrices containing $N(n)$ elements. In this note we give a simpler derivation of Schur's results and an extension of these results from algebraically closed fields to arbitrary fields.

If $A_1, \cdots, A_{N(n)}$ is a set of linearly independent commutative matrices, the set $\mathcal{U}$ of matrices $\sum A_i \phi_i$ where $\phi_i$ is arbitrary in the underlying field $\Phi$ is a commutative subalgebra containing the identity of the matrix algebra $\Phi_n$. Hence $N(n)$ is the maximal dimensionality of commutative subalgebras of $\Phi_n$. It is easy to see that $N(n) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$. For consider the set $\mathcal{B}_n$ of matrices

$$
\begin{pmatrix}
0 & A \\
0 & 0
\end{pmatrix}
$$

where if $n = 2\nu$, $A$ is arbitrary in $\Phi$, and if $n = 2\nu - 1$, $A$ is an arbitrary matrix of $\nu$ rows and $\nu - 1$ columns. Thus $\dim \mathcal{B}_n = \left\lfloor \frac{n^2}{4} \right\rfloor$. It may be verified that $\mathcal{B}_n$ is a zero algebra. Hence the algebra $\mathcal{B}_n$ obtained by adjoining 1 to $\mathcal{B}_n$ is a commutative algebra of dimensionality $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$. We remark also that if $n = 2\nu - 1$ we may replace $\mathcal{B}_n$ by the algebra $\overline{\mathcal{B}}_n$ of matrices of the form (1) in which $A$ is an arbitrary matrix of $\nu - 1$ rows and $\nu$ columns. We denote by $\overline{\mathcal{B}}_n$ the extension of $\mathcal{B}_n$ obtained by adjoining 1.

To prove that $N(n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ it suffices to assume that $\Phi$ is algebraically closed. For if $A_1, \cdots, A_{N(n)}$ are linearly independent and commutative in $\Phi_n$, then they have these properties in $\Sigma_n$ for any extension field $\Sigma$ of the field $\Phi$. Thus $N(n, \Phi) \leq N(n, \Sigma)$. We shall therefore assume that $\Phi$ is algebraically closed. Let $\mathcal{U}$ be a commutative subalgebra of $\Phi_n$ containing the identity and let $N$ be the dimensionality of $\mathcal{U}$ over $\Phi$. We suppose first that $\mathcal{U}$ is an indecomposable algebra of matrices. Then it is known that by replacing $\mathcal{U}$ by a similar set we may suppose that the matrices of $\mathcal{U}$ have the form

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Thus \( \mathfrak{N} = (1) + \mathfrak{N} \) where \( \mathfrak{N} \) is a nilpotent algebra of matrices in proper triangular form, that is, of the form (2) in which \( \alpha = 0 \). Evidently \( \dim \mathfrak{N} = N - 1 \).

Let the \( k_1 \)th column (\( k_1 > 1 \)) be the first column for which there exists a matrix \( U_{1k_1} \) in \( \mathfrak{N} \) with element in the \((1, k_1)\) position not equal to 0. We may suppose that the element in the \((1, k_1)\) position of \( U_{1k_1} \) is 1. We normalize \( U_{1k_1} \) further by using the following lemma.

**Lemma 1.** Let \( U \in \Phi_n \) and let \( V \) be the matrix obtained from \( U \) by adding the \( k \)th column multiplied by \( \theta \) to the \( l \)th column (\( k \neq l \)) and then subtracting the \( l \)th row multiplied by \( \theta \) from the \( k \)th row. Then \( U \) and \( V \) are similar.

We have \( V = S^{-1}US \) where \( S = 1 + e_{kl} \theta, e_{kl} \) the matrix with 1 in the \((k, l)\) position and 0's elsewhere.

We may apply this lemma to \( U_{1k_1} \) and replace it by a matrix whose first row is \( e_{k_1} = (0, \ldots, 1, 0, \ldots, 0) \) where the 1 is in the \( k_1 \)th column. The operations required for this purpose are additions of multiples of the \( k_1 \)th column to later columns and additions to the \( k_1 \)th row of later rows. These operations replace \( \mathfrak{N} \) by a properly triangular set of matrices \( \mathfrak{N}' \) similar to \( \mathfrak{N} \) such that all the elements in the \((1, j)\) position with \( j < k_1 \) in \( \mathfrak{N}' \) are 0 and such that \( \mathfrak{N}' \) contains a matrix \( V_{1k_1} \) (similar to \( U_{1k_1} \)) whose first row is \( e_{k_1} \). Now let \( \Psi' \) be the subspace of \( \mathfrak{N}' \) of matrices in which the elements in the \((1, k_1)\) position are 0 and suppose that the \( k_2 \)th column (\( k_2 > k_1 \)) is the first column for which there is a matrix \( U_{1k_2} \) in \( \mathfrak{N}' \) with element in the \((1, k_2)\) place not equal to 0. Evidently any matrix in \( \mathfrak{N}' \) has the form \( V_{1k_1} \beta_1 + P', P' \) in \( \Psi' \). We now apply to \( U_{1k_2} \) the process used before for \( U_{1k_1} \), and replace it by a matrix \( V_{1k_2} \) similar to it and having \( e_{k_2} \) for first row. The set \( \mathfrak{N}' \) will be transformed into a set \( \mathfrak{N}'' \) of properly triangular matrices and \( V_{1k_1} \) changed into a new matrix which we shall again denote as \( V_{1k_1} \) with first row \( e_{k_1} \). Any matrix in \( \mathfrak{N}'' \) has the form \( A = V_{1k_1} \beta_1 + P'', P'' \) in \( \Psi'' \), the transform of the set \( \Psi' \). It is clear that the elements in the \((1, j)\) position, \( j < k_2 \), for any matrix in \( \Psi'' \) are 0. Hence \( A = V_{1k_1} \beta_1 + V_{1k_2} \beta_2 + S'' \) where \( S'' \) is in the subspace \( \mathfrak{S}'' \) of \( \mathfrak{N}'' \) of matrices having 0 in the \((1, j)\) position with \( j \leq k_2 \). This process may be continued and proves the following lemma.

**Lemma 2.** The set \( \mathfrak{N} \) is similar to a set \( \mathfrak{N}^{(r)} \) of properly triangular
matrices that contain matrices $V_{1k_1}, \ldots, V_{1k_r}$ such that the first row of $V_{1k_i}$ is $e_{k_i}, 1 < k_1 < k_2 < \cdots < k_r$, and such that any matrix in $\mathfrak{N}^{(r)}$ has the form $\sum V_{1k_i} \beta_i + Z$, where $Z$ has first row 0.

Now let $\mathfrak{N}_2$ be the subset of $\mathfrak{N}^{(r)}$ of matrices $Z$ having first row 0. Evidently $\mathfrak{N}^{(r)} = \{ V_{1k_1}, \ldots, V_{1k_r} \} + \mathfrak{N}_2$ and the $V_{1k_i}$ are linearly independent. Hence $\dim \mathfrak{N}^{(r)} = N - 1 = r + \dim \mathfrak{N}_2$. Now we note that if $Z \in \mathfrak{N}_2$, the first row of $V_{1k_i}Z$ is the $k_i$th row of $Z$ and the first row of $Z V_{1k_i}$ is 0. Hence the $k_i$th row of every matrix $Z$ in $\mathfrak{N}_2$ is 0.

We now repeat the argument for $\mathfrak{N}_2$. Then $\mathfrak{N}_2$ may be replaced by a set $\mathfrak{N}_2^{(o)}$ similar to $\mathfrak{N}_2$ such that (1) $\mathfrak{N}_2^{(o)}$ is properly triangular, (2) $\mathfrak{N}_2^{(o)}$ contains matrices $V_{2t_1}, \ldots, V_{2t_r}$ having first row 0 and second row $e_{t_1}, \ldots, e_{t_r}$ respectively, such that any matrix in $\mathfrak{N}_2^{(o)}$ has the form $\sum V_{2t_i} \beta_i + Z$ where $Z$ is a matrix with first two rows 0. Let $\mathfrak{N}_3$ denote the set of matrices $Z$. We assert that if $s = l_1$ or $s = k_j$ then the $s$th row of $\mathfrak{N}_3$ is 0. This is clear if $s = l_i$. Hence suppose that $s = k_j \neq$ any $l_i$. Then the matrices of $\mathfrak{N}_3$ all have $k_j$th row 0 and the operations performed in passing from $\mathfrak{N}_2$ to $\mathfrak{N}_2^{(o)}$ do not affect this row. Hence the $k_j$th row of every matrix in $\mathfrak{N}_2^{(o)}$ is 0. Evidently $N - 1 = r + s + \dim \mathfrak{N}_3$.

We now write $k_1 = k_{11}, l_i = k_{2i}, r = r_1, s = r_2$. Then if we continue this process we see that $N - 1$ is equal to the number of matrices in the following set

$$\begin{align*}
  & e_{1k_{11}}, \ldots, e_{1k_{r_1}} \\
  & e_{2k_{11}}, \ldots, e_{2k_{r_2}} \\
  & \hspace{1cm} \ldots \\
\end{align*}$$

(3) where $1 < k_{11} < \cdots < k_{1r_1}, 2 < k_{21} < k_{22} < \cdots < k_{2r_2}, \ldots$, and $r_i = 0$ if $i = k_{ji}$ with $j < i$. Let $s_1, s_2, \ldots, s_m$ be the complete set of integers $k_{ij}$ arranged in increasing order. Then it is clear that $N - 1 \leq N(s_1, s_2, \ldots, s_m)$, the number of matrices in the set

$$\begin{align*}
  & e_{1s_1}, e_{2s_1}, \ldots, e_{1s_1 - 1}s_1 \\
  & e_{2s_1}, e_{2s_2}, \ldots, e_{2s_1 - 1}s_{21}, e_{2s_1 + 1}s_{21}, \ldots, e_{2s_1 - 1}s_2 \\
  & \hspace{1cm} \ldots \\
\end{align*}$$

(4) Evidently

$$\begin{align*}
  N(s_1, s_2, \ldots, s_m) &= (s_1 - 1) + (s_2 - 2) + \cdots + (s_m - m) \\
  &= \sum s_i - m(m + 1)/2.
\end{align*}$$

Hence we have
\(N - 1 \leq N(s_1, \ldots, s_m) \leq N(n - m + 1, \ldots, n)\)

\[= m(n - m).\]

Now \(m(n - m)\) attains its maximum value for \(m = \lfloor n/2 \rfloor\). If \(n = 2\nu\) this maximum is \(\nu^2\) and if \(n = 2\nu - 1\), it is \(\nu(\nu - 1)\). Thus the maximum value is \([n^2/4]\). This proves for indecomposable algebras \(\mathcal{A}\) the following theorem.

**Theorem 1.** If \(\mathcal{A}\) is a commutative subalgebra of \(\Phi_n\), \(\dim \mathcal{A} \leq [n^2/4] + 1\).

If \(\mathcal{A}\) is decomposable we suppose that the matrices of \(\mathcal{A}\) have the form

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\]

where \(A \in \Phi_{n_1}\) and \(B \in \Phi_{n_2}\), \(n_i \geq 1\), \(n_1 + n_2 = n\). We may assume that the theorem holds for the \(\Phi_{n_i}\).

Case 1. \(n = 2\nu - 1\), \(n_1 = 2\nu_1 - 1\), \(n_2 = 2\nu_2\). Here \(\nu = \nu_1 + \nu_2\) and \(N \leq \nu_1(\nu_1 - 1) + 1 + \nu_2^2 + 1 \leq \nu(\nu - 1) + 1\). Equality holds between the last two terms only when \(n = 3\).

Case 2. \(n = 2\nu\), \(n_1 = 2\nu_1 - 1\), \(n_2 = 2\nu_2 - 1\). Here \(\nu = \nu_1 + \nu_2 - 1\) and \(N \leq \nu_1(\nu_1 - 1) + 1 + \nu_2(\nu_2 - 1) + 1 \leq \nu^2 + 1\). Equality holds only if \(n = 2\).

Case 3. \(n = 2\nu\), \(n_1 = 2\nu_1\), \(n_2 = 2\nu_2\). Here \(\nu = \nu_1 + \nu_2\) and \(N = \nu_1^2 + 1 + \nu_2^2 + 1 < \nu^2 + 1\). Thus the theorem is proved.

We have also proved the following theorem.

**Theorem 2.** The maximum number \(N(n)\) of linearly independent commutative matrices of \(n\) rows and columns is given by the formula

\(N(n) = [n^2/4] + 1\).

We shall investigate next the form of commutative subalgebras \(\mathcal{A}\) of \(\Phi_n\) of the maximum dimensionality \(N(n)\). Suppose first that \(\mathcal{A}\) has the structure \(\mathcal{A} = (1) + \mathcal{R}\) where \(\mathcal{R}\) is a nilpotent algebra. Then it is known that by replacing \(\mathcal{R}\) by a similar set we may suppose that the matrices of \(\mathcal{R}\) are properly triangular. We may apply the above considerations to \(\mathcal{R}\). By (3), (4), (5) and (6) we see that if \(n = 2\nu\) we must have \(k_{11} = k_{21} = \cdots = k_{11} = \nu + 1\), \(\cdots\), \(k_{1\nu} = k_{2\nu} = \cdots = k_{\nu\nu} = n\) as the set of \(k\)'s in (3). If \(n = 2\nu - 1\) the set of \(k\)'s is either \(k_{11} = \cdots = k_{1\nu} = \nu + 1\), \(\cdots\), \(k_{1\nu - 1} = \cdots = k_{\nu - 1, \nu - 1} = n\) or \(k_{11} = \cdots = k_{\nu - 1, \nu - 1} = \nu\), \(\cdots\), \(k_{11} = \cdots = k_{\nu - 1, \nu} = n\). Suppose first that \(n\) is even. Let \(\mathcal{R}^{(r)} (r = \nu)\) and \(\mathcal{R}_2\) be determined as before. It is clear that \(\mathcal{R}^{(r)}\) is similar to \(\mathcal{R}\) by a matrix in \(\Phi_n\) and we need not assume here that \(\Phi\) is algebraically closed. The matrices of \(\mathcal{R}_2\) have the form
Since $k_{31} = \nu + 1$ it is clear that the second row of $R$ is 0. Moreover the operations used to pass from $\mathfrak{N}_2$ to $\mathfrak{N}_3$ affect only the last $\nu$ rows and last $\nu$ columns of $\mathfrak{N}_2$. Hence the third row of $R$ is the same as the third row of the corresponding matrix in $\mathfrak{N}_3$. Since $k_{31} = \nu + 1$ the third row of $R$ is 0. Similarly the other rows of $R$ are 0, and $R = 0$ in (7). Now $\dim \mathfrak{N}_2 = \nu^2 - \nu$. Hence $\mathfrak{N}_2$ consists of all matrices of the form (7) in which $R = 0$ and $A$ is arbitrary. Let

$$V_{1j} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & V_j & & \vdots & \\ & & & & T_j & & & \\ \end{pmatrix}, \quad j = \nu + 1, \ldots, n,$$

where the 1 is in the $j$th column and $T_j$ is a properly triangular matrix.

Since $V_{1j}B = BV_{1j}$ the following holds in $\Phi$: 

$$\begin{pmatrix} 0 & \cdots & 0 \\ & A \\ \end{pmatrix} T_j = 0.$$

Since $A$ is arbitrary, $T_j = 0$. Thus $\mathfrak{N}^{(r)}$ is the set $\mathfrak{B}_n$ and $\mathfrak{A}$ is similar to the algebra $\mathfrak{B}_n$ defined before. If $n$ is odd a similar argument shows that $\mathfrak{A}$ is similar either to $\mathfrak{B}_n$ or to $\mathfrak{B}_n$.

We suppose now that $\mathfrak{A}$ is arbitrary. Evidently $\mathfrak{A}$ contains the identity matrix. Since $n > 3$ by the proof of Theorem 1, $\mathfrak{A}$ is indecomposable. Moreover if $\Omega$ is the algebraic closure of $\Phi$ then $\mathfrak{A}_\Omega$ is an indecomposable algebra containing the identity. It follows that $\mathfrak{A}_\Omega$ is similar to a set of matrices of the form (1). Hence $\mathfrak{A}_\Omega = (1) + \mathfrak{B}$ where $\mathfrak{B}$ is nilpotent and so $\mathfrak{A}_\Omega$ is similar to either $\mathfrak{B}_n(\Omega)$ or $\mathfrak{B}_n(\Omega)$. Thus $\mathfrak{B}$ is a zero algebra. Now let $\mathfrak{N}$ be the radical of the algebra $\mathfrak{A}$ and consider the semi-simple algebra $\mathfrak{A} = \mathfrak{A} - \mathfrak{N}$. The extension $\mathfrak{A}_\Omega$ is a homomorphic image of $\mathfrak{A}_\Omega$. Hence $\mathfrak{A}_\Omega = (1) + \mathfrak{B}$ where $\mathfrak{B}$ is a zero algebra. The structure of $\mathfrak{A}$ is given by the following lemma.

**Lemma 3.** If $\mathfrak{A}$ is a semi-simple commutative algebra such that $\mathfrak{A}_n = (1) + \mathfrak{B}$ where $\mathfrak{B}$ is a zero algebra, then either $\mathfrak{A} = (1)$ or $\Phi$ is an imperfect field of characteristic 2 and $\mathfrak{A} = \Phi(x)$ where $x^2 = \xi$, a non-square in $\Phi$.

Since $\mathfrak{A}$ is semi-simple, $\mathfrak{A}$ is a direct sum of fields, but since $\mathfrak{A}_\Omega$ has only one idempotent element, $\mathfrak{A}$ is a field. Let $\mathfrak{A} > (1)$. Then $\mathfrak{A}$ has no
separable subfields, for if \( \Sigma \) were such a subfield \( \Sigma_0 \) is a direct sum of fields and \( \Lambda_\sigma \) would contain more than one idempotent element. Thus \( \Phi \) has characteristic \( p \neq 0 \) and \( \Lambda_\sigma \) contains an element \( x \) such that \( x^p = \xi \) is in \( \Phi \) where \( \xi \) is not a \( p \)th power in \( \Phi \). Now there exists an element \( \eta \) in \( \Omega \) such that \( \eta^p = \xi \) and hence the element \( z = x - \eta \) in \( \Lambda_\sigma \) is nilpotent of index \( p \). Since \( \Lambda_\sigma \) is a zero algebra, \( p = 2 \). It follows readily that in this case \( \Lambda = \Phi(x), x^2 = \xi \).

This lemma shows that unless \( \Phi \) is an imperfect field of characteristic 2 any commutative subalgebra \( \Omega \) of \( \Phi_n(n > 3) \) of maximum dimensionality has a difference algebra with respect to its radical \( \mathcal{R} \) of dimensionality 1. Since \( \Lambda \) contains the identity, \( \Lambda = (1) + \mathcal{R} \). As we have seen, this implies that \( \Lambda \) is similar to either \( \mathcal{B}_n \) or to \( \mathcal{B}_n \).

**Theorem 3.** Suppose that \( \Phi \) is not an imperfect field of characteristic 2 and let \( n > 3 \). Then if \( \Omega \) is a subalgebra of \( \Phi_n \) of maximum dimensionality \( N(n) \), \( \Omega \) is similar to \( \mathcal{B}_n \) if \( n = 2v \) and \( \Omega \) is similar to either \( \mathcal{B}_n \) or \( \mathcal{B}_n \) if \( n = 2v - 1 \).

As a consequence we have the following theorem.

**Theorem 4.** Let \( \Phi, n \) and \( \Lambda \) be as in Theorem 3. Then \( \Lambda = (1) + \mathcal{R} \) where \( \mathcal{R} \) is a zero algebra.

We remark finally that if \( n \) is odd the sets \( \mathcal{B}_n \) and \( \mathcal{B}_n \) are not similar. This may be seen by considering the sets \( \mathcal{B}_n \) and \( \mathcal{B}_n \). Let \( \mathcal{S} = \mathcal{S}(\mathcal{E}) \) be the space determined by the columns of the matrices of \( \mathcal{B}_n(\mathcal{B}_n) \). Then \( \dim \mathcal{S} = v \) and \( \dim \mathcal{E} = v - 1 \). On the other hand if \( \mathcal{B}_n \) were similar to \( \mathcal{B}_n \) we would have \( \dim \mathcal{S} = \dim \mathcal{E} \). It follows that \( \mathcal{B}_n \) and \( \mathcal{B}_n \) are not similar and hence \( \mathcal{B}_n \) and \( \mathcal{B}_n \) are not similar. Thus in this case there are for \( n = 2v - 1 > 3 \) two distinct classes in the sense of similarity of commutative subalgebras of dimensionality \( N(n) \).

Johns Hopkins University

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\(^3\) If \( n = 2, 3, \mathcal{R} \) may be decomposable. The determination of these algebras is readily obtained.