# SCHUR'S THEOREMS ON COMMUTATIVE MATRICES

N. JACOBSON

In 1905 I. Schur[1] proved that the maximum number $N(n)$ of linearly independent commutative matrices of $n$ rows and columns is given by the formula $N(n)=[n^2/4]+1=\nu^2+1$ if $n=2\nu$ and $=\nu(\nu-1)+1$ if $n=2\nu-1$. Schur also determined the sets of linearly independent commutative matrices containing $N(n)$ elements. In this note we give a simpler derivation of Schur's results and an extension of these results from algebraically closed fields to arbitrary fields.

If $A_1, \cdots, A_{N(n)}$ is a set of linearly independent commutative matrices, the set $\mathfrak{A}$ of matrices $\sum A_i\phi_i$ where $\phi_i$ is arbitrary in the underlying field $\Phi$ is a commutative subalgebra containing the identity of the matrix algebra $\Phi_n$. Hence $N(n)$ is the maximal dimensionality of commutative subalgebras of $\Phi_n$. It is easy to see that $N(n)\geqq [n^2/4]+1$. For consider the set $\mathfrak{Z}_n$ of matrices

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \tag{1}$$

where if $n=2\nu$, $A$ is arbitrary in $\Phi_\nu$ and if $n=2\nu-1$, $A$ is an arbitrary matrix of $\nu$ rows and $\nu-1$ columns. Thus $\dim \mathfrak{Z}_n=[n^2/4]$. It may be verified that $\mathfrak{Z}_n$ is a zero algebra. Hence the algebra $\mathfrak{B}_n$ obtained by adjoining 1 to $\mathfrak{Z}_n$ is a commutative algebra of dimensionality $[n^2/4]+1$. We remark also that if $n=2\nu-1$ we may replace $\mathfrak{Z}_n$ by the algebra $\overline{\mathfrak{Z}}_n$ of matrices of the form (1) in which $A$ is an arbitrary matrix of $\nu-1$ rows and $\nu$ columns. We denote by $\overline{\mathfrak{B}}_n$ the extension of $\overline{\mathfrak{Z}}_n$ obtained by adjoining 1.

To prove that $N(n)\leqq[n^2/4]+1$ it suffices to assume that $\Phi$ is algebraically closed. For if $A_1, \cdots, A_{N(n)}$ are linearly independent and commutative in $\Phi_n$, then they have these properties in $\Sigma_n$ for any extension field $\Sigma$ of the field $\Phi$. Thus $N(n, \Phi)\leqq N(n, \Sigma)$. We shall therefore assume that $\Phi$ is algebraically closed. Let $\mathfrak{A}$ be a commutative subalgebra of $\Phi_n$ containing the identity and let $N$ be the dimensionality of $\mathfrak{A}$ over $\Phi$. We suppose first that $\mathfrak{A}$ is an indecomposable algebra of matrices. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{A}$ have the form

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[1] *Zur Theorie vertauschbaren Matrizen*, J. Reine Angew. Math. vol. 130 (1905) pp. 66–76.

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(2) \qquad \begin{pmatrix} \alpha & & & * \\ & \alpha & & \\ & & \ddots & \\ 0 & & & \alpha \end{pmatrix}.
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Thus $\mathfrak{A} = (1) + \mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent algebra of matrices in proper triangular form, that is, of the form (2) in which $\alpha = 0$. Evidently $\dim \mathfrak{N} = N-1$.

Let the $k_1$th column ($k_1 > 1$) be the first column for which there exists a matrix $U_{1k_1}$ in $\mathfrak{N}$ with element in the $(1, k_1)$ position not equal to 0. We may suppose that the element in the $(1, k_1)$ position of $U_{1k_1}$ is 1. We normalize $U_{1k_1}$ further by using the following lemma.

Lemma 1. *Let $U \in \Phi_n$ and let $V$ be the matrix obtained from $U$ by adding the kth column multiplied by $\theta$ to the lth column ($k \neq l$) and then subtracting the lth row multiplied by $\theta$ from the kth row. Then $U$ and $V$ are similar.*

We have $V = S^{-1}US$ where $S = 1 + e_{kl}\theta$, $e_{kl}$ the matrix with 1 in the $(k, l)$ position and 0's elsewhere.

We may apply this lemma to $U_{1k_1}$ and replace it by a matrix whose first row is $e_{k_1} = (0, \cdots, 1, 0, \cdots, 0)$ where the 1 is in the $k_1$th column. The operations required for this purpose are additions of multiples of the $k_1$th column to later columns and additions to the $k_1$th row of later rows. These operations replace $\mathfrak{N}$ by a properly triangular set of matrices $\mathfrak{N}'$ similar to $\mathfrak{N}$ such that all the elements in the $(1, j)$ position with $j < k_1$ in $\mathfrak{N}'$ are 0 and such that $\mathfrak{N}'$ contains a matrix $V_{1k_1}$ (similar to $U_{1k_1}$) whose first row is $e_{k_1}$. Now let $\mathfrak{P}'$ be the subspace of $\mathfrak{N}'$ of matrices in which the elements in the $(1, k_1)$ position are 0 and suppose that the $k_2$th column ($k_2 > k_1$) is the first column for which there is a matrix $U_{1k_2}$ in $\mathfrak{P}'$ with element in the $(1, k_2)$ place not equal to 0. Evidently any matrix in $\mathfrak{N}'$ has the form $V_{1k_1}\beta_1 + P'$, $P'$ in $\mathfrak{P}'$. We now apply to $U_{1k_2}$ the process used before for $U_{1k_1}$ and replace it by a matrix $V_{1k_2}$ similar to it and having $e_{k_2}$ for first row. The set $\mathfrak{N}'$ will be transformed into a set $\mathfrak{N}''$ of properly triangular matrices and $V_{1k_1}$ changed into a new matrix which we shall again denote as $V_{1k_1}$ with first row $e_{k_1}$. Any matrix in $\mathfrak{N}''$ has the form $A = V_{1k_1}\beta_1 + P''$, $P''$ in $\mathfrak{P}''$, the transform of the set $\mathfrak{P}'$. It is clear that the elements in the $(1, j)$ position, $j < k_2$, for any matrix in $\mathfrak{P}''$ are 0. Hence $A = V_{1k_1}\beta_1 + V_{1k_2}\beta_2 + S''$ where $S''$ is in the subspace $\mathfrak{S}''$ of $\mathfrak{N}''$ of matrices having 0 in the $(1, j)$ position with $j \leq k_2$. This process may be continued and proves the following lemma.

Lemma 2. *The set $\mathfrak{N}$ is similar to a set $\mathfrak{N}^{(r)}$ of properly triangular* *matrices that contain matrices* $V_{1k_1}, \cdots, V_{1k_r}$ *such that the first row of* $V_{1k_i}$ is $e_{k_i}$, $1<k_1<k_2< \cdots <k_r$, *and such that any matrix in* $\mathfrak{N}^{(r)}$ *has the form* $\sum V_{1k_i}\beta_i+Z$, *where* $Z$ *has first row* 0.

Now let $\mathfrak{N}_2$ be the subset of $\mathfrak{N}^{(r)}$ of matrices $Z$ having first row 0. Evidently $\mathfrak{N}^{(r)}=\{V_{1k_1}, \cdots, V_{1k_r}\}+\mathfrak{N}_2$ and the $V_{1k_i}$ are linearly independent. Hence dim $\mathfrak{N}^{(r)}=N-1=r+\dim \mathfrak{N}_2$. Now we note that if $Z\in\mathfrak{N}_2$, the first row of $V_{1k_i}Z$ is the $k_i$th row of $Z$ and the first row of $ZV_{1k_i}$ is 0. Hence the $k_i$th row of every matrix $Z$ in $\mathfrak{N}_2$ is 0.

We now repeat the argument for $\mathfrak{N}_2$. Then $\mathfrak{N}_2$ may be replaced by a set $\mathfrak{N}_2^{(s)}$ similar to $\mathfrak{N}_2$ such that (1) $\mathfrak{N}_2^{(s)}$ is properly triangular, (2) $\mathfrak{N}_2^{(s)}$ contains matrices $V_{2l_1}, \cdots, V_{2l_s}$ having first row 0 and second row $e_{l_1}, \cdots, e_{l_s}$, respectively, such that any matrix in $\mathfrak{N}_2^{(s)}$ has the form $\sum V_{2l_i}\beta_i+Z$ where $Z$ is a matrix with first two rows 0. Let $\mathfrak{N}_3$ denote the set of matrices $Z$. We assert that if $s=l_i$ or $s=k_j$ then the $s$th row of $\mathfrak{N}_3$ is 0. This is clear if $s=l_i$. Hence suppose that $s=k_j\neq$any $l_i$. Then the matrices of $\mathfrak{N}_2$ all have $k_j$th row 0 and the operations performed in passing from $\mathfrak{N}_2$ to $\mathfrak{N}_2^{(s)}$ do not affect this row. Hence the $k_j$th row of every matrix in $\mathfrak{N}_2^{(s)}$ is 0. Evidently $N-1=r+s+\dim \mathfrak{N}_3$.

We now write $k_i=k_{1i}$, $l_i=k_{2i}$, $r=r_1$, $s=r_2$. Then if we continue this process we see that $N-1$ is equal to the number of matrices in the following set

$$(3)\qquad \begin{array}{c} e_{1k_{11}}, \cdots, e_{1k_1r_1} \\ e_{2k_{21}}, \cdots, e_{2k_2r_2} \\ \cdots\cdots \end{array}$$

where $1<k_{11}< \cdots <k_{1r_1}$, $2<k_{21}<k_{22}< \cdots <k_{2r_2}, \cdots$, and $r_i=0$ if $i=k_{jl}$ with $j<i$. Let $s_1, s_2, \cdots, s_m$ be the complete set of integers $k_{ij}$ arranged in increasing order. Then it is clear that $N-1 \leq N(s_1, s_2, \cdots, s_m)$, the number of matrices in the set

$$(4)\qquad \begin{array}{l} e_{1s_1}, e_{2s_1}, \cdots, e_{s_1-1,s_1} \\ e_{1s_2}, e_{2s_2}, \cdots, e_{s_1-1,s_2}, e_{s_1+1,s_2}, \cdots, e_{s_2-1,s_2} \\ \cdots\cdots\cdots\cdots\cdots . \end{array}$$

Evidently

$$(5)\qquad \begin{aligned} N(s_1, s_2, \cdots, s_m) &= (s_1-1)+(s_2-2)+\cdots+(s_m-m) \\ &= \sum s_i - m(m+1)/2. \end{aligned}$$

Hence we have

$$N - 1 \leqq N(s_1, \cdots, s_m) \leqq N(n - m + 1, \cdots, n) \\ = m(n - m). \tag{6}$$

Now $m(n-m)$ attains its maximum value for $m=[n/2]$. If $n=2\nu$ this maximum is $\nu^2$ and if $n=2\nu-1$, it is $\nu(\nu-1)$. Thus the maximum value is $[n^2/4]$. This proves for indecomposable algebras $\mathfrak{A}$ the following theorem.

THEOREM 1. *If $\mathfrak{A}$ is a commutative subalgebra of $\Phi_n$,* dim $\mathfrak{A} \leqq [n^2/4] +1$.

If $\mathfrak{A}$ is decomposable we suppose that the matrices of $\mathfrak{A}$ have the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in \Phi_{n_1}$ and $B \in \Phi_{n_2}$, $n_i \geqq 1$, $n_1+n_2=n$. We may assume that the theorem holds for the $\Phi_{n_i}$.

Case 1. $n=2\nu-1$, $n_1=2\nu_1-1$, $n_2=2\nu_2$. Here $\nu=\nu_1+\nu_2$ and $N \leqq \nu_1(\nu_1-1)+1+\nu_2^2+1 \leqq \nu(\nu-1)+1$. Equality holds between the last two terms only when $n=3$.

Case 2. $n=2\nu$, $n_1=2\nu_1-1$, $n_2=2\nu_2-1$. Here $\nu=\nu_1+\nu_2-1$ and $N \leqq \nu_1(\nu_1-1)+1+\nu_2(\nu_2-1)+1 \leqq \nu^2+1$. Equality holds only if $n=2$.

Case 3. $n=2\nu$, $n_1=2\nu_1$, $n_2=2\nu_2$. Here $\nu=\nu_1+\nu_2$ and $N=\nu_1^2+1+\nu_2^2+1<\nu^2+1$. Thus the theorem is proved.

We have also proved the following theorem.

THEOREM 2. *The maximum number $N(n)$ of linearly independent commutative matrices of $n$ rows and columns is given by the formula* $N(n)=[n^2/4]+1$.

We shall investigate next the form of commutative subalgebras $\mathfrak{A}$ of $\Phi_n$ of the maximum dimensionality $N(n)$. Suppose first that $\mathfrak{A}$ has the structure $\mathfrak{A}=(1)+\mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent algebra. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{N}$ are properly triangular. We may apply the above considerations to $\mathfrak{N}$. By (3), (4), (5) and (6) we see that if $n=2\nu$ we must have $k_{11}=k_{21}= \cdots =k_{\nu 1}=\nu+1, \cdots, k_{1\nu}=k_{2\nu}= \cdots =k_{\nu\nu}=n$ as the set of $k$'s in (3). If $n=2\nu-1$ the set of $k$'s is either $k_{11}= \cdots =k_{\nu 1} =\nu+1, \cdots, k_{1\,\nu-1}= \cdots =k_{\nu\,\nu-1}=n$ or $k_{11}= \cdots =k_{\nu-1\,1}=\nu, \cdots, k_{1\nu}= \cdots =k_{\nu-1\,\nu}=n$. Suppose first that $n$ is even. Let $\mathfrak{N}^{(r)}$ $(r=\nu)$ and $\mathfrak{N}_2$ be determined as before. It is clear that $\mathfrak{N}^{(r)}$ is similar to $\mathfrak{N}$ by a matrix in $\Phi_n$ and we need not assume here that $\Phi$ is algebraically closed. The matrices of $\mathfrak{N}_2$ have the form

$$B = \left(\begin{array}{c|c} 0 \cdots 0 & \overbrace{0 \cdots 0}^{\nu} \\ R & A \\ \hline 0 & 0 \end{array}\right)\Big\}\nu. \tag{7}$$

Since $k_{21} = \nu + 1$ it is clear that the second row of $R$ is 0. Moreover the operations used to pass from $\mathfrak{N}_2$ to $\mathfrak{N}_3$ affect only the last $\nu$ rows and last $\nu$ columns of $\mathfrak{N}_2$. Hence the third row of $R$ is the same as the third row of the corresponding matrix in $\mathfrak{N}_3$. Since $k_{31} = \nu + 1$ the third row of $R$ is 0. Similarly the other rows of $R$ are 0, and $R = 0$ in (7). Now $\dim \mathfrak{N}_2 = \nu^2 - \nu$. Hence $\mathfrak{N}_2$ consists of all matrices of the form (7) in which $R = 0$ and $A$ is arbitrary. Let

$$V_{1j} = \left(\begin{array}{c|c} 0 & 0 \cdots 0 \;\; 1 \;\; 0 \cdots 0 \\ & V_j \\ \hline & T_j \end{array}\right), \qquad j = \nu + 1, \cdots, n,$$

where the 1 is in the $j$th column and $T_j$ is a properly triangular matrix. Since $V_{1j}B = BV_{1j}$ the following holds in $\Phi_\nu$:

$$\begin{pmatrix} 0 \cdots 0 \\ A \end{pmatrix} T_j = 0.$$

Since $A$ is arbitrary, $T_j = 0$. Thus $\mathfrak{N}^{(r)}$ is the set $\mathfrak{Z}_n$ and $\mathfrak{A}$ is similar to the algebra $\mathfrak{B}_n$ defined before. If $n$ is odd a similar argument shows that $\mathfrak{A}$ is similar either to $\mathfrak{B}_n$ or to $\overline{\mathfrak{B}}_n$.

We suppose now that $\mathfrak{A}$ is arbitrary. Evidently $\mathfrak{A}$ contains the identity matrix. Since $n > 3$ by the proof of Theorem 1, $\mathfrak{A}$ is indecomposable. Moreover if $\Omega$ is the algebraic closure of $\Phi$ then $\mathfrak{A}_\Omega$ is an indecomposable algebra containing the identity. It follows that $\mathfrak{A}_\Omega$ is similar to a set of matrices of the form (1). Hence $\mathfrak{A}_\Omega = (1) + \mathfrak{Z}$ where $\mathfrak{Z}$ is nilpotent and so $\mathfrak{A}_\Omega$ is similar to either $\mathfrak{B}_n(\Omega)$ or $\overline{\mathfrak{B}}_n(\Omega)$. Thus $\mathfrak{Z}$ is a zero algebra. Now let $\mathfrak{N}$ be the radical of the algebra $\mathfrak{A}$ and consider the semi-simple algebra $\overline{\mathfrak{A}} = \mathfrak{A} - \mathfrak{N}$. The extension $\overline{\mathfrak{A}}_\Omega$ is a homomorphic image of $\mathfrak{A}_\Omega$. Hence $\overline{\mathfrak{A}}_\Omega = (1) + \overline{\mathfrak{Z}}$ where $\overline{\mathfrak{Z}}$ is a zero algebra. The structure of $\overline{\mathfrak{A}}$ is given by the following lemma.

LEMMA 3. *If $\overline{\mathfrak{A}}$ is a semi-simple commutative algebra such that $\overline{\mathfrak{A}}_\Omega = (1) + \overline{\mathfrak{Z}}$ where $\overline{\mathfrak{Z}}$ is a zero algebra, then either $\overline{\mathfrak{A}} = (1)$ or $\Phi$ is an imperfect field of characteristic* 2 *and $\overline{\mathfrak{A}} = \Phi(x)$ where $x^2 = \xi$, a non-square in $\Phi$.*

Since $\mathfrak{A}$ is semi-simple, $\overline{\mathfrak{A}}$ is a direct sum of fields, but since $\overline{\mathfrak{A}}_\Omega$ has only one idempotent element, $\overline{\mathfrak{A}}$ is a field. Let $\overline{\mathfrak{A}} > (1)$. Then $\overline{\mathfrak{A}}$ has no separable subfields, for if $\Sigma$ were such a subfield $\Sigma_\Omega$ is a direct sum of fields and $\overline{\mathfrak{A}}_\Omega$ would contain more than one idempotent element. Thus $\Phi$ has characteristic $p \neq 0$ and $\overline{\mathfrak{A}}$ contains an element $x$ such that $x^p = \xi$ is in $\Phi$ where $\xi$ is not a $p$th power in $\Phi$. Now there exists an element $\eta$ in $\Omega$ such that $\eta^p = \xi$ and hence the element $z = x - \eta$ in $\overline{\mathfrak{A}}_\Omega$ is nilpotent of index $p$. Since $\overline{\mathfrak{Z}}$ is a zero algebra, $p = 2$. It follows readily that in this case $\mathfrak{A} = \Phi(x)$, $x^2 = \xi$.

This lemma shows that unless $\Phi$ is an imperfect field of characteristic 2 any commutative subalgebra $\mathfrak{A}$ of $\Phi_n (n > 3)$ of maximum dimensionality has a difference algebra with respect to its radical $\mathfrak{N}$ of dimensionality 1. Since $\mathfrak{A}$ contains the identity, $\mathfrak{A} = (1) + \mathfrak{N}$. As we have seen, this implies that $\mathfrak{A}$ is similar to either $\mathfrak{B}_n$ or to $\overline{\mathfrak{B}}_n$.

THEOREM 3. *Suppose that $\Phi$ is not an imperfect field of characteristic 2 and let $n > 3$. Then if $\mathfrak{A}$ is a subalgebra of $\Phi_n$ of maximum dimensionality $N(n)$, $\mathfrak{A}$ is similar to $\mathfrak{B}_n$ if $n = 2\nu$ and $\mathfrak{A}$ is similar to either $\mathfrak{B}_n$ or $\overline{\mathfrak{B}}_n$ if $n = 2\nu - 1$.*[2]

As a consequence we have the following theorem.

THEOREM 4. *Let $\Phi$, $n$ and $\mathfrak{A}$ be as in Theorem 3. Then $\mathfrak{A} = (1) + \mathfrak{N}$ where $\mathfrak{N}$ is a zero algebra.*

We remark finally that if $n$ is odd the sets $\mathfrak{B}_n$ and $\overline{\mathfrak{B}}_n$ are not similar. This may be seen by considering the sets $\mathfrak{Z}_n$ and $\overline{\mathfrak{Z}}_n$. Let $\mathfrak{S}(\overline{\mathfrak{S}})$ be the space determined by the columns of the matrices of $\mathfrak{Z}_n(\overline{\mathfrak{Z}}_n)$. Then $\dim \mathfrak{S} = \nu$ and $\dim \overline{\mathfrak{S}} = \nu - 1$. On the other hand if $\mathfrak{Z}_n$ were similar to $\overline{\mathfrak{Z}}_n$ we would have $\dim \mathfrak{S} = \dim \overline{\mathfrak{S}}$. It follows that $\mathfrak{Z}_n$ and $\overline{\mathfrak{Z}}_n$ are not similar and hence $\mathfrak{B}_n$ and $\overline{\mathfrak{B}}_n$ are not similar. Thus in this case there are for $n = 2\nu - 1 > 3$ two distinct classes in the sense of similarity of commutative subalgebras of dimensionality $N(n)$.

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[2] If $n = 2, 3$, $\mathfrak{A}$ may be decomposable. The determination of these algebras is readily obtained.