ON LINEAR EQUATIONS IN HILBERT SPACE

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Given an infinite matrix \( A = \|a_{ij}\| \) where \( a_{ij} \) is complex and

\[
\sum_{j=1}^{\infty} |a_{ij}|^2 < + \infty, \quad i = 1, 2, \ldots ,
\]

the problem of solving the system of linear equations

\[
y_i = \sum_{j=1}^{\infty} a_{ij}x_j, \quad i = 1, 2, \ldots ,
\]

has been studied from several points of view. For arbitrary \( y \),
E. Schmidt\(^1\) has given necessary and sufficient conditions on the \( a_{ij} \),
\( y \), so that the system (2) have a solution \( x = (x_j) \in H_2 \) (Hilbert space).
Schmidt shows that if a solution exists, the solution of minimum norm
is unique, and gives explicit formulas for this solution. If \( A \) defines
a linear transformation \( T \) on \( H_2 \) to \( H_2 \), F. Riesz\(^2\) gives necessary and
sufficient conditions that an inverse \( T^{-1} \) exist, that is, that the solution
\( x = T^{-1}(y) \) where \( T^{-1} \) is a linear transformation. The following
problem stands between these two: Find conditions on the elements
of \( A \) so that the system (2) have a solution \( x \in H_2 \) for each \( y \in H_2 \).
Such conditions will permit the use of Schmidt’s formulas to express
the minimal solution \( x \) for each \( y \) but this of course does not imply
the existence of an inverse of the matrix \( A \). We give a solution of this
problem by a method which depends on a property, which seems new,
of the \( m \)-rowed minors of the matrices \( A_{i_1 \ldots i_m} = \|a_{i_1 j_1}\|_{1 \leq i_1 \leq \ldots \leq i_m} \) and on
Cramer’s rule.

Let

\[
a(i_1, \ldots , i_m; j_1, \ldots , j_m) = \det \|a_{i_1 j_1}\|_{1 \leq i_1 \leq \ldots \leq i_m}
\]

be the determinant of the columns \( j_1, \ldots , j_m \) of \( A_{i_1 \ldots i_m} \). If \( B = \|b_{ij}\| \)
satisfies (1) and \( B'_{i_1 \ldots i_m} \) is the transposed of \( B_{i_1 \ldots i_m} \), the determinant
\( \det A_{i_1 \ldots i_m}B'_{i_1 \ldots i_m} = \det \left\| \sum_{k=1}^{\infty} a_{i_k b_{k i}} \right\|_{1 \leq i_1 \leq \ldots \leq i_m} \) is finite. Because of the
continuity of a determinant as a function of its elements

\[
\det A_{i_1 \ldots i_m}B'_{i_1 \ldots i_m} = \lim_n \det \left\| \sum_{k=1}^{n} a_{i_k b_{k i}} \right\|_{1 \leq i_1 \leq \ldots \leq i_m}
\]

\[
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\]


\(^2\) F. Riesz, Les systèmes d'équations linéaires a une infinité d'inconnues, Paris, 1913, p. 86.
There is a theorem on the minors of products of square matrices which, with slight modification in its proof, yields the identity

\[ \det \left| \sum_{k=1}^{n} a_{i,k} b_{i,k} \right|_{1 \leq s, t \leq m} = \sum_{\{i_1, \cdots, i_m\}} a(i_1, \cdots, i_m; j_1, \cdots, j_m) b(i_1, \cdots, i_m; j_1, \cdots, j_m), \]

where the sum is extended over all combinations \( j_1, \cdots, j_m \) in \( 1, \cdots, n \).

**Theorem 1.** If \( A, B \) satisfy (1), then

\[ \det A_{i_1 \cdots i_m} B_{i_1' \cdots i_m'} = \sum_{\{i_1, \cdots, i_m\}} a(i_1, \cdots, i_m; j_1, \cdots, j_m) b(i_1, \cdots, i_m; j_1, \cdots, j_m), \]

where the sum is extended over all combinations of positive integers \( j_1, \cdots, j_m \). The series converges absolutely and

\[ | \det A_{i_1 \cdots i_m} B_{i_1' \cdots i_m'} | \leq [\det A_{i_1 \cdots i_m} A_{i_1' \cdots i_m'}]^{1/2} [\det B_{i_1 \cdots i_m} B_{i_1' \cdots i_m'}]^{1/2}. \]

**Proof.** By Schwartz’ inequality we have, from (4) and (3) with \( B = A \),

\[ \sum_{\{i_1, \cdots, i_m\}} | a(i_1, \cdots, i_m; j_1, \cdots, j_m) b(i_1, \cdots, i_m; j_1, \cdots, j_m) | \leq \left[ \sum_{\{i_1, \cdots, i_m\}} | a(i_1, \cdots, i_m; j_1, \cdots, j_m) |^2 \right]^{1/2} \left[ \sum_{\{i_1, \cdots, i_m\}} | b(i_1, \cdots, i_m; j_1, \cdots, j_m) |^2 \right]^{1/2} \]

\[ \leq [\det A_{i_1 \cdots i_m} A_{i_1' \cdots i_m'}]^{1/2} [\det B_{i_1 \cdots i_m} B_{i_1' \cdots i_m'}]^{1/2}, \quad n \geq m. \]

The conclusion is now evident as a consequence of (3).

If we define

\[ | A B' | = \lim_{m \to \infty} \sup_{i_1, \cdots, i_m} \sup_{j_1, \cdots, j_m} \left| \sum_{\{i_1, \cdots, i_m\}} a(i_1, \cdots, i_m; j_1, \cdots, j_m) x \times b(i_1, \cdots, i_m; j_1, \cdots, j_m) \right| \]

we have a Schwarz inequality for matrices:

\[ |AB'| \leq |A\overline{A}'|^{1/2} |B\overline{B}'|^{1/2}. \]

The following lemma contains the Gram condition for linear dependence.

**Lemma 1.** The following statements are equivalent:

(a) The rows of \(A_1 \ldots m\) are linearly dependent.
(b) \(\text{Det } A_1 \ldots m \overline{A}'_1 \ldots m = 0\).
(c) All \(m\)-rowed minors of \(A_1 \ldots m\) equal zero.

**Proof.** That (a) implies (c) is immediate. The equivalence of (b) and (c) follows from Theorem 1 with \(A = B\). It remains to show that (c) implies (a). This is evident if \(m = 1\). Assuming this statement for \(m - 1\), it is true for \(m\) if all the \((m - 1)\)-rowed minors of \(A_{m-1}\) vanish. If one such minor does not vanish, say

\[ \det \|a_{ij}\|_{1 \leq i, j \leq m-1} \neq 0, \]

we denote by \(c_k\) the cofactor of \(a_{km}\) in the determinant of the first \(m\) columns of \(A_1 \ldots m\). Then \(c_m \neq 0\) and

\[ \sum_{k=1}^{m} c_k a_{kj} = 0, \quad j = 1, \ldots, m - 1. \]

But this sum vanishes for all other values of \(j\) because of (c). Hence (c) implies (a).

**Theorem 2.** If \(A\) satisfies (1), the finite system

\[ y_i = \sum_{j=1}^{\infty} a_{ij} x_j, \quad i = 1, \ldots, m, \]

has a solution \(x \in H_s\) for each \(y_1, \ldots, y_m\) if and only if

\[ \det A_1 \ldots m \overline{A}'_1 \ldots m \neq 0. \]

**Proof.** The necessity is a consequence of Lemma 1. If the condition is satisfied, then there is a nonvanishing \(a(1, \ldots, m; j_1, \ldots, j_m)\) by (c) of Lemma 1 and a solution \(x = (x_j)\) where \(x_j = 0\) for \(j \neq j_1, \ldots, j_m\) and \(x_{j_1}, \ldots, x_{j_m}\) are determined by Cramer's rule.

**Corollary.** If \(A\) satisfies (1) and the system (2) has a solution \(x \in H_s\) for each \(y \in H_s\), then \(\det A_1 \ldots m \overline{A}'_1 \ldots m \neq 0\) for all \(m\).

An estimate of the minimum norm of the solution of the finite system (5) may be given in terms of a series of finite minors in \(A_{i_1} \ldots i_m\).
Let \( J = [j_1, \ldots, j_m] \) be a combination of \( m \) distinct positive integers and let \( S_m \) be a set of \( J \) such that no two \( J \)'s have a common integer while every positive integer is in some \( J \in S_m \). Let

\[
\alpha_{i_1 \ldots i_m} = \left[ \sup_{S_m} \sum_{J \in S_m} |a(i_1, \ldots, i_m; j_1, \ldots, j_m)|^2 \right]^{1/2},
\]
\[
\sigma_{i_1 \ldots i_m} = \left[ \sum_{[j_1 \ldots j_m]} |a(i_1, \ldots, i_m; j_1, \ldots, j_m)|^2 \right]^{1/2}.
\]

Since \( S_m \) is a subset of the sum of all \( j_1, \ldots, j_m \) we have the following lemma.

**Lemma 2.** \( \alpha_{i_1 \ldots i_m} \leq \sigma_{i_1 \ldots i_m} \),

\[
\sup_{S_m} \sum_{J \in S_m} \sum_{j=1}^m |a(1, \ldots, k-1, k+1, \ldots, m; j_1, \ldots, j_{k-1}j_{k+1}, \ldots, j_m)|^2 \leq \sum_{[j_1 \ldots j_m]} |a(1, \ldots, k-1, k+1, \ldots, m; j_1, \ldots, j_{m-1})|^2.
\]

**Theorem 3.** If \( A \) satisfies (1) and the finite system (5) has a solution \( x^m \in H_k \) for each \( y^m = (y_1, \ldots, y_m, 0, 0, \ldots) \), then

\[
\inf \|x^m\| \leq \|y^m\| \left[ \sum_{k=1}^m \frac{|a_{1, \ldots, k_1, \ldots, k_{m-1}m}|^2}{\alpha_{1, \ldots, m}} \right]^{1/2}.
\]

**Proof.** By Theorem 2, \( \det A_{i_1 \ldots i_m} = 0 \) and so \( \alpha_{1, \ldots, m} = 0 \). Let \( M_{k_1}^{m_1} \) be the cofactor of \( a_{k_1} \) in \( a(1, \ldots, m; j_1, \ldots, j_m) \). The system (5) has a solution \( x_{j_1 \ldots j_m} \) defined by

\[
x_{j_1 \ldots j_m} = \begin{cases} 
\sum_{k=1}^m y_k M_{k_1}^{m_1}/a(1, \ldots, m; j_1, \ldots, j_m), & j = j_1, \ldots, j_m, \\
0, & j \neq j_1, \ldots, j_m.
\end{cases}
\]

We have

\[
\|x_{j_1 \ldots j_m}\|^2 = |a(i_1, \ldots, i_m; j_1 \ldots j_m)|^2 \leq \sum_{j=1}^m \sum_{k=1}^m |y_k M_{k_1}^{m_1}|^2 \leq \|y^m\|^2 \sum_{k=1}^m \sum_{j=1}^m |M_{k_1}^{m_1}|^2 \leq \|y^m\|^2 \sum_{k=1}^m \sum_{j=1}^m |a(1, \ldots, k-1, k+1, \ldots, m; j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_m)|^2.
\]

Hence
\[
\inf \|x\|^{2} \leq \|y\|^{2} \sum_{k=1}^{m} \sup_{S_{m}} \sum_{i=1}^{m} |a(1, \ldots, k - 1, \\
k + 1, \ldots, m; j_{1} \cdots j_{k-j_{k+1}} \cdots j_{m})|^{2} \leq \|y\|^{2} \sum_{k=1}^{m} \frac{a_{k+1, \ldots, m}}{a_{1, \ldots, m}} \cdot |j_{1} \cdots j_{m}|^{2/3}
\]

by Lemma 2. The conclusion follows at once.

A sufficient condition for the solution of the system (2) for each \(y \in H\) may be obtained by restricting the constants

\[
\alpha_{m} = \left( \sum_{k=1}^{m} \frac{a_{1, \ldots, k-1, k+1, \ldots, m}}{a_{1, \ldots, m}} \right)^{1/2}, \quad m = 1, 2, \ldots.
\]

**Theorem 4.** If \(A\) satisfies (1), its rows are linearly independent, and \(\alpha = \lim \inf_{m} \alpha_{m} < + \infty\), then for each \(y \in H\) the system (2) has a solution \(x \in H_{2}\) such that

\[
\|x\| \leq \alpha \|y\|.
\]

**Proof.** Consider any \(y \in H_{2}\) and any \(\epsilon > 0\). The sequence contains a subsequence \(\alpha_{m_{n}} < \alpha + \epsilon\). From Theorem 3 it follows that for each \(\mu\) there is an \(x^{\mu} = (x^{\mu}_{i}) \in H_{2}\) such that

\[
y_{i} = \sum_{j=1}^{\infty} a_{ij}x_{j}, \quad i = 1, \ldots, m_{\mu},
\]

\[
\|x^{\mu}\| \leq (\alpha + \epsilon)\|y\|.
\]

Applying a diagonal process to \((x^{\mu}_{j})\) one finds a subsequence \(x^{\nu} = (x^{\nu}_{j})\) and an \(x = (x_{j}) \in H_{2}\) such that

\[
\lim_{\nu} x^{\nu}_{j} = x_{j}, \quad j = 1, 2, \ldots,
\]

\[
\|x\| \leq (\alpha + \epsilon)\|y\|.
\]

Since for all \(\nu, N > 0\) and \(1 \leq i \leq m_{\nu}\)

\[
y_{i} - \sum_{j=1}^{\infty} a_{ij}x_{j} = \sum_{j=1}^{N} a_{ij}(x_{j}^{\nu} - x_{j}) + \sum_{j=N+1}^{\infty} a_{ij}x_{j}^{\nu} - \sum_{j=N+1}^{\infty} a_{ij}x_{j},
\]

\(x\) solves the system (1).

Now consider \(\epsilon_{n} \downarrow 0\). For each \(\mu\) there is an \(x^{\mu} \in H_{2}\) which solves the system (2) and such that \(\|x^{\mu}\| \leq (\alpha + \epsilon_{\mu})\|y\|\). Repeating the diagonal process and the above argument, one finds an \(x \in H_{2}\) such that \(\|x\| \leq \alpha \|y\|\) and which solves the system (2).