TWO PROPERTIES OF THE FUNCTION $\cos x$

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The function $f(x) = A \cos n(x + B)$, where $A$, $B$ are any real constants and $n$ is an integer, has the properties:

(I) $f(x)$ is real valued for all real $x$, of period $2\pi$, and continuous.

(II) $f(x)$ is differentiable, and there exist constants $a$, $b$ such that, for all $x$,
$$f'(x) = af(x + b).$$

(III) Given any constants $a$, $a'$, $b$, $b'$, there exist constants $c$, $d$ such that, for all $x$,
$$af(x + a') + bf(x + b') = cf(x + d).$$

The object of this note is to show that, conversely, any function $f(x)$ which has property (I) and either (II) or (III) is necessarily of the form $f(x) = A \cos n(x + B)$. The latter result is used to derive the parallelogram law of addition of forces from certain other basic assumptions.

**Theorem 1.** Let $f(x)$ have properties (I) and (II). Then there exist constants $A$, $B$ and an integer $n$ such that $f(x) = A \cos n(x + B)$.

**Proof.** It follows from (II) that $f(x)$ is of class $C^\infty$ and hence, from (I), can be represented by a convergent Fourier series, which, moreover, may be differentiated termwise. Thus for some complex constants $k_n$,

\begin{equation}
    f(x) = \sum k_n e^{inx}, \quad f'(x) = \sum in k_n e^{inx},
\end{equation}

\begin{equation}
    f'(x) - af(x + b) = \sum k_n (in - ae^{inb}) e^{inx}.
\end{equation}

It follows from (II) that for every integer $n$,
\begin{equation}
    k_n (in - ae^{inb}) = 0.
\end{equation}

If $f(x) \equiv 0$ then the theorem is trivial. Otherwise, there will exist an $n$ for which $k_n \neq 0$. It follows that
\begin{equation}
    in = ae^{inb}.
\end{equation}

Taking absolute values we have
\begin{equation}
    n = \pm a.
\end{equation}

Thus there can be at most two values of $n$ for which $k_n \neq 0$, and these

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values are negatives of one another. Thus for some integer $n$,

$$f(x) = k_{-n}e^{-inx} + k_ne^{inx}.$$  

Since, by (I), $f(x)$ is real valued, it follows that $k_{-n}$ and $k_n$ are complex conjugates, and the proof is complete.

**Theorem 2.** Let $f(x)$ have properties (I) and (III). Then there exist constants $A$, $B$ and an integer $n$ such that $f(x) = A\cos n(x+B)$.

**Proof.** Since, by (I), $f(x)$ is continuous and of period $2\pi$, it possesses at least a formal Fourier series,\(^1\)

$$f(x) \sim \sum k_ne^{inx}, \quad k_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int}dt.$$  

By (III), there exist constants $c, d$ such that the function

$$g(x) = f(x+1) + f(x) - cf(x+d),$$

continuous and of period $2\pi$, is identically zero. (This is the only consequence of (III) that we shall use.) Hence

$$0 \sim \sum k_n(e^{in} + 1 - ce^{ind})e^{inx}.$$  

It follows that for every $n$,

$$k_n(e^{in} + 1 - ce^{ind}) = 0.$$  

If $f(x) \equiv 0$ then the theorem is trivial. Otherwise, there will exist an $n$ for which $k_n \neq 0$. For any such $n$ it follows from (9) that

$$e^{in} + 1 = ce^{ind}.$$  

Taking absolute values and squaring, it follows that

$$\cos n = (c^2 - 2)/2.$$  

Hence if $m$ and $n$ are any two integers for which $k_m, k_n \neq 0$, it follows from (11) that $\cos m = \cos n$. Hence for some integer $r$,

$$m = \pm n + 2\pi r.$$  

Since $\pi$ is irrational, it follows that $r = 0$ and $m = \pm n$. Thus the formal Fourier series for $f(x)$ consists of only two terms,

$$f(x) \sim k_{-n}e^{-inx} + k_ne^{inx}.$$  

But in this case, since the functions $e^{inx}$ are complete with respect

\(^1\) The proof given here follows a suggestion of Paul R. Halmos. The author's original proof required the unnecessary assumption that $f(x)$ be of class $C^1$.\]
to continuous functions, the relation \( \sim \) can be replaced by an identity,

\[
(14) \quad f(x) = k_-n e^{-inx} + k_n e^{inx}.
\]

Since \( f(x) \) is real valued, \( k_-n \) and \( k_n \) must be complex conjugates, and the theorem is proved.

We shall now apply Theorem 2 to derive the law of addition of forces. For simplicity, let us consider only forces acting at a fixed point in a fixed plane in which the angular coordinate \( x \) is defined. With such a force we identify the real valued function \( F(x) \) which specifies the scalar component of the force in the direction \( x \); thus a force is represented by a real valued function of period \( 2\pi \). By the sum of two forces \( F_1(x) \) and \( F_2(x) \) we mean the function \( F_1(x) + F_2(x) \). Our assumptions are the following:

(i) All forces are geometrically similar. By this we mean that there exists a fixed function \( f(x) \) of period \( 2\pi \) such that any force \( F(x) \) can be written in the form

\[
(15) \quad F(x) = A_F \cdot f(x + \alpha_F),
\]

where \( A_F \) and \( \alpha_F \) are constants determined by \( F(x) \). We need not assume that all values of the constants \( A_F \) and \( \alpha_F \) can occur in (15), but we shall assume that there exist at least the forces \( F_1(x) = f(x) \) and \( F_2(x) = f(x+1) \).

(ii) The sum of two forces is a force. Together with (i), this implies that the function \( f(x) \) has the property that for certain constants \( c, d \) and for every \( x \),

\[
(16) \quad f(x + 1) + f(x) = cf(x + d).
\]

(iii) The function \( f(x) \) is continuous, non-constant, and vanishes for at most two values in the interval \( 0 \leq x < 2\pi \).

The proof of Theorem 2 shows that the function \( f(x) \), continuous, real valued, of period \( 2\pi \), and satisfying (16), must be of the form

\[
(17) \quad f(x) = A \cos n(x + B),
\]

where \( n \) is an integer. The hypotheses of (iii) ensure that \( n \) can be chosen as 1. The parallelogram law of addition of forces is an immediate consequence.

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