1. Preliminary definitions. By an algebra, we shall mean below any collection $A$ of elements, combined by any set of single-valued operations $f_a$,

$$y = f_a(x_1, \ldots, x_{n(a)}).$$

The number of distinct operations (that is, the range of the variable $a$) may be infinite, but for our main result (Theorem 2), we shall require every $n(a)$ to be finite—that is, it will concern algebras with finitary operations.

The concepts of subalgebra, congruence relation on an algebra, homomorphism of one algebra $A$ onto (or into) another algebra with the same operations, and of the direct union $A_1 \times \cdots \times A_r$, of any finite or infinite class of algebras with the same operations have been developed elsewhere.¹ More or less trivial arguments establish a many-one correspondence between the congruence relations $\theta_i$ on an algebra $A$ and the homomorphic images $H_i = \theta_i(A)$ of the algebra (isomorphic images being identified); moreover the congruence relations on $A$ form a lattice (the structure lattice of $A$). In this lattice, the equality relation will be denoted 0; all other congruence relations will be called proper.

More or less trivial arguments also show (cf. Lattice theory, Theorem 3.20) that the isomorphic representations of any algebra $A$ as a subdirect union, or subalgebra $S \leq H_1 \times \cdots \times H_r$ of a direct union of algebras $H_i$, correspond essentially one-one to the sets of congruence relations $\theta_i$ on $A$ such that $A\theta_i = 0$. In fact, given such a set of $\theta_i$, the correspondence

$$\theta: a \rightarrow [\theta_1(a), \ldots, \theta_r(a)] = [h_1, \ldots, h_r]$$

exhibits the desired isomorphism of $A$ with a subalgebra of $H_1 \times \cdots \times H_r$, where $H_i = \theta_i(A)$. Incidentally, the number of $S_i$ can

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be infinite. What is more important, the operations of $A$ need not even be finitary.\footnote{This is observed in N. H. McCoy and Deane Montgomery, \textit{A representation of generalized Boolean rings}, Duke Math. J. vol. 3 (1937) p. 46, line 12.}

Equally trivial arguments extend a well known theorem of Emmy Noether on commutative rings\footnote{Cf. van der Waerden, \textit{Moderne Algebra}, first ed., vol. 2, p. 36. The unicity theorem on p. 40 does not apply to abstract algebras in general, however.} to abstract algebras in general. In order to state this extension, we first define an algebra $A$ to be \textit{subdirectly irreducible} if in any finite or infinite representation (2), some $\theta_i$ is itself an isomorphism. This means that the meet $\theta^*$ of all proper congruence relations on $A$ is itself a proper congruence relation. In lattice-theoretic language, it means that the structure lattice $L(A)$ of $A$ contains a point $\theta^*>0$ such that $\theta>0$ implies $\theta \geq \theta^*$. Hence if $A$ is subdirectly irreducible, $\theta \cap \theta' = 0$ in $L(A)$ implies $\theta = 0$ or $\theta' = 0$; such an $A$ we shall call \textit{weakly irreducible}. If $L(A)$ satisfies the descending chain condition, and $A$ is weakly irreducible, then it is evidently also subdirectly irreducible in the strong sense.

If $L(A)$ satisfies the \textit{ascending} chain condition, then it is evident by induction\footnotemark[2] that there exists a representation of 0 as the meet $0 = \theta_0 \cap \cdots \cap \theta_r$ of a finite number of irreducible elements. This yields the following easy generalization of Emmy Noether's theorem on commutative rings.

\textbf{Theorem 1.} Any algebra $A$ whose structure lattice satisfies the ascending chain condition is isomorphic with a subdirect union of a finite number of weakly irreducible algebras.

For this result, we still do not need to assume that $A$ has finitary operations.

\textbf{2. Main theorem.} Our principal result is the partial extension of Theorem 1 to algebras \textit{without} chain condition. As will be seen in §3, this result will contain as special cases many known theorems and some new theorems.

\textbf{Theorem 2.} Any algebra $A$ with finitary operations is isomorphic with a subdirect union of subdirectly irreducible algebras.

\textbf{Proof.} For any $a \neq b$ of $A$, consider the set $S(a, b)$ of all congruence relations $\theta$ on $A$, such that $a \neq b \pmod{\theta}$. If $T$ is any linearly ordered subset of $S(a, b)$, the union $\tau$ of the $\theta \in T$ is defined by the rule

\begin{equation}
    x \equiv y \pmod{\tau} \text{ means } x \equiv y \pmod{\theta} \text{ for some } \theta \in T.
\end{equation}

It is evident that $a \neq b \pmod{\tau}$, and that if $A$ has finitary operations,
then \( \tau \) is a congruence relation. Hence, in the structure lattice \( L(A) \) of \( A \), the union of any linearly ordered subset of \( S(a, b) \) exists and is in \( S(a, b) \). But this is the first hypothesis of the "first form" of Zorn's Lemma.\(^4\) The conclusion is that \( S(a, b) \) contains a maximal element, \( \theta_{a,b} \). We next consider \( H_{a,b} \), the homomorphic image of \( A \), mod \( \theta_{a,b} \).

Every proper congruence relation \( \theta > 0 \) corresponds to a \( \theta' > \theta_{a,b} \); and since \( \theta_{a,b} \) is maximal in \( S(a, b) \), this implies \( a \equiv b \) (mod \( \theta' \)). Hence the meet \( \theta^* \) of the \( \theta > 0 \) in \( H_{a,b} \), defined by

\[
x \equiv y \pmod{\theta^*} \text{ means } x \equiv y \pmod{\theta} \text{ for all } \theta > 0,
\]

will satisfy \( a \equiv b \) (mod \( \theta^* \)), and hence \( \theta^* > 0 \). Hence (cf. §1) \( H_{a,b} \) is subdirectly irreducible.

Finally, the meet of all \( \theta_{a,b} \) is 0, since we have identically \( x \not\equiv y \pmod{\theta_{a,y}} \). Hence, by the theorem cited in §1, paragraph 3, \( A \) is isomorphic with a subdirect union of the (subdirectly irreducible) \( H_{a,b} \), q.e.d.

3. Applications. Theorem 2 has importance mainly because subdirectly irreducible algebras may be specifically described in so many cases.

**Lemma 1.** A weakly irreducible distributive lattice or Boolean algebra must consist of 0 and I alone.

**Proof for Distributive Lattices.** For any \( a \), the endomorphisms \( \theta_a : x \rightarrow x \cup a \) and \( \theta_a' : x \rightarrow x \cap a \) have the property\(^6\) that \( \theta_a \cap \theta_a' = 0 \), and neither defines the equality relation unless \( a = 0 \) or \( a = I \).

**Proof for Boolean Algebras.** Let \( x \equiv y \pmod{\theta_a} \) mean \( |x - y| \leq a \) (symmetric difference notation); then \( \theta_a \cap \theta_{a'} = 0 \), and neither defines the equality relation unless \( a = 0 \) or \( a' = 0 \) (\( a = I \)).

**Corollary 1.** Any distributive lattice is isomorphic with a ring of sets.\(^7\)

**Corollary 2.** Any Boolean algebra is isomorphic with a field of sets.\(^7\)

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\(^6\) We omit discussing the obvious isomorphism between \( L(H_{a,b}) \) and the sublattice of \( \theta^* > \theta_{a,b} \) in \( L(A) \).

LEMMA 2. A subdirectly irreducible commutative ring \( R \) without nilpotent elements is a field. \(^8\)

PROOF. As in §1, \( R \) will have a unique minimal ideal (that is, congruence relation) \( J \). But for any \( a \neq 0 \) in \( J \), since \( aa \neq 0 \), \( aJ > 0 \). Moreover since \( (aJ)R = a(JR) \subseteq aJ \), \( aJ \) is an ideal, \( 0 < aJ \leq J \). Consequently \( aJ = J \)—whence \( J \) is a field (Huntington's postulates) with unit \( e \). The set \( 0:e \) of all \( x \in R \) such that \( ex = 0 \) is an ideal, and \( (0:e) \cap J = 0 \) by what we have just shown; hence \( 0:e = 0 \). But for any \( x \in R \), \( e(x-ex) = 0 \); hence \( (x-ex) \in 0:e = 0 \), and \( x = (x-ex) + ex \), \( 0 = ex \in J \). We conclude that \( R = J \) is a field, q.e.d.

One might infer that by Theorem 2 any commutative ring without nilpotent elements was a subdirect product of fields, but this reasoning would be invalid. It is not necessarily true that a homomorphic image of rings without nilpotent elements is itself without nilpotent elements.

On the other hand, any homomorphic image of a \( p \)-ring (or commutative ring in which \( a^p = a \) for some prime \( p \)) is itself a \( p \)-ring, and evidently without nilpotent elements, since \( a^{pn} = a \) for all \( n \). Furthermore, a field in which \( a^p = a \) can contain only \( p \) elements, and must be \( GF(p) \) (or the "field" \( 0 \)).

COROLLARY 3. Any \( p \)-ring is a subdirect union of \( GF(p) \), or consists of \( 0 \) alone. \(^7\)

Again, any homomorphic image of a regular ring in the sense of von Neumann (or ring in which any \( a \) has a "relative inverse" \( u \) such that \( auu = a \)) is itself regular, and evidently without nilpotent elements if commutative (since \( a^n u^{n-1} = auuu \cdots uu = a \neq 0 \)).

COROLLARY 4. Any commutative "regular" ring is a subdirect union of fields. \(^7\)

If one were interested in obtaining corollaries of Theorem 1, one might show that even a weakly irreducible \( p \)-ring or regular ring was a field. Again, one might show (van der Waerden, op. cit. p. 32) that, in a weakly irreducible commutative ring satisfying the chain condition, every divisor of zero is nilpotent; this would yield E. Noether's theorem that every commutative ring satisfying the chain condition was a subdirect union of a finite number of primary rings.

Similarly, one can show easily that the only weakly irreducible vector space over a field \( F \) is the one-dimensional vector space \( V(F; 1) \) (or \( 0 \)). It follows that any vector space is a subdirect union of one-

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\(^8\) This lemma was suggested to the author in conversation by N. H. McCoy.
dimensional vector spaces. Actually (due to the existence of bases) a stronger result is well known.

**Lemma 3.** The only weakly irreducible commutative groups $G$ are the "generalized cyclic" groups: the additive subgroups of the rationals, and those of the rationals mod one.

We omit the proof, which follows easily from the fact that a commutative group with two generators is cyclic unless it contains two disjoint subgroups (the latter hypothesis would make $G$ weakly reducible).

**Corollary 5.** Any commutative group is a subdirect union of generalized cyclic groups.

The center of any weakly irreducible hypercentral (alias nilpotent) group $H$ is generalized cyclic (the proof is trivial, granted Lemma 3); the converse also holds if $H$ is finite. Hence we have the following corollary.

**Corollary 6.** Any hypercentral group is a subdirect union of groups with generalized cyclic centers.

Further, any weakly irreducible commutative $l$-group (lattice-ordered group) is known to be simply ordered. This yields the following corollary.

**Corollary 7.** Any commutative $l$-group is a subdirect union of simply ordered $1$-groups.

One can easily show (although we omit the proof) that any closed element in a closure algebra (in the sense of McKinsey and Tarski) determines a congruence relation, essentially through relativization with respect to the complementary open set. Then from the definition of well-connectedness one obtains the following corollary.

**Corollary 8.** Any "closure algebra" is a subdirect union of "well-connected" closure algebras.

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11 *The algebra of topology*, Ann. of Math. vol. 45 (1944) pp. 141–191. The definition of well-connectedness is on p. 147; the concept of relativization is developed on p. 151.