For by Proposition 4, $FA = A$ for every $A$ and $A (A^{-1} F) = EF = F$.
Let $S_1, \ldots, S_n$ be $n$ statements. Let $A_i$ be the statement: "All the preceding statements are annulled but $S_i$ is true." It is interesting to note that the statements $A_i$ form an idempotent $(l, r)$ system.

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A CONJECTURE IN ELEMENTARY NUMBER THEORY

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A well known conjecture of Catalan states that if $f(n)$ is the sum of all divisors of $n$ except $n$, then the sequence of iterates of $f(n)$ is either eventually periodic or ends at 1. It not only seems impossible to prove this, but it is also very difficult to verify.\(^1\)

Another conjecture of Poulet,\(^2\) which appears equally difficult to prove, has the doubtful merit that it is easy to verify. Let $\sigma(n)$ be the sum of all divisors of $n$, and let $\phi(n)$ be Euler's function. Then for any integer $n$ the sequence

$$f_0(n) = n, \quad f_{2k+1}(n) = \sigma(f_{2k}(n)), \quad f_{2k}(n) = \phi(f_{2k-1}(n))$$

is eventually periodic.

We have verified this conjecture to $n=10000$ (extending Poulet's verification) by using Glaisher's tables.\(^3\) The checking was facilitated by the following observation: if the conjecture is to be checked for all $n < x$, it is enough to find a member of the sequence other than the first which is less than $x$.

The longest cycle found was in the sequence $f_{18}(9216)$. It starts with $f_1(9216)$, and is: 34560, 122640, 27648, 81800, 30976, 67963, 54432, 183456, 48384, 163520, 55296, 163800, 34560. However our method of checking does not show that this is the largest cycle up to 10000, and in fact Poulet found that $f_{18}(1800)$ has the same length 12.

As a rule $\phi(\sigma(n))$ is less than $n$. In fact, it can be shown that for every $\epsilon > 0$, $\phi(\sigma(n)) < en$, except for a set of density 0. The proof follows from the following two observations:

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\(^1\) L. E. Dickson, Theorems and tables on the sum of the divisors of a number, Quart. J. Math. vol. 44 (1913) pp. 264–296, and P. Poulet, La Chasse aux Nombres, vol. 1, pp. 68–72, and vol. 2, p. 188.

\(^2\) P. Poulet, Nouvelles suites arithmétiques, Sphinx vol. 2 (1932) pp. 53–54.

\(^3\) J. W. L. Glaisher, Number-divisor tables, British Association for the Advancement of Science, Mathematical Tables, vol. 8.
For a given prime $p$, the set of all $n$ such that $\sigma(n) \equiv 0 \pmod{p}$ is of density $1$. The set of all integers not divisible by any prime $q$ of the form $pq - 1$ is of density zero, since $\sum_{q \geq 1} 1/q$ diverges. Hence the set of all integers divisible by a prime $q > N$ of this type is of density $1$. But the set of all integers divisible by $q^2$, $q > N$, is of density less than $\sum_{q > N} 1/q^2 = o(1)$. Therefore, if $x$ is large, the number of $n$ less than $x$ such that $\sigma(n) \equiv 0 \pmod{p}$ exceeds $(1 - \epsilon)x$.

Except for $\epsilon x$ integers $n$ less than $x$, $\sigma(n) < c(\epsilon)n$.

This follows from the fact that $\sum_{n < x} \sigma(n) \sim \pi^2 n^2/12$.

Choose $p$ so that $\prod_{q \equiv 0 \pmod{p}} (1 - 1/q) < \delta/c(\epsilon)$. Then, if $x$ is sufficiently large, all except $\eta x + \epsilon x$ integers $n$ less than $x$ have $\sigma(n) < c(\epsilon)n$, $\sigma(n) \equiv 0 \pmod{q}$ for all $q \leq p$. But, with these exceptions, $\phi[\sigma(n)] < \delta n$, which completes the proof, since $\eta$ and $\epsilon$ are arbitrary.

In much the same way it can be shown that for every $c > 0$, $\sigma[\sigma(n)] > cn$ except for a set of density zero.

Actually, much more can be shown. Except for a set of density zero, $e^{-\gamma} \sigma[\sigma(n)] \log \log \log n \sim \sigma(n)$, and $e^{-\gamma} \sigma[\phi(n)]/\log \log \log n \sim \phi(n)$, where $\gamma$ is Euler's constant. The proof is suppressed, but it might be noted that the reason for this result is that, for almost all $n$, $\phi(n)$ and $\sigma(n)$ are both divisible by all primes less than $(\log \log n)^{1+\epsilon}$, and by relatively few primes greater than $(\log \log n)^{1+\epsilon}$.

There exist numbers for which $\phi(\sigma(n)) = n$. Up to 2500 these numbers are 1, 2, 8, 12, 128, 240, 720; while two further solutions are $2^{18}$ and $2^{81}$. Poulet gives many others; we do not know whether there are infinitely many solutions.

We state two further conjectures:

(a) Form the sequence $\sigma(n)$, $\sigma(\sigma(n))$, $\phi(\sigma(\sigma(n)))$, $\sigma(\phi(\sigma(\sigma(n))))$, in which the functions are successively applied in the order $\sigma$, $\sigma$, $\phi$, $\sigma$, $\sigma$, $\phi$, ..., This sequence seems to tend to infinity if $n$ is large enough.

(b) On the other hand, the sequence $\phi(n)$, $\phi(\phi(n))$, $\sigma(\phi(\phi(n)))$, ..., in which the order is $\phi$, $\phi$, $\sigma$, $\phi$, $\phi$, $\sigma$, $\phi$, ..., seems to converge to 1, for all $n$.

Obviously many more such conjectures can be formulated.