ORBIT-CLOSURE DECOMPOSITIONS AND ALMOST PERIODIC PROPERTIES

W. H. GOTTSCALK

Let $X$ be a metric space with metric $\rho$, let $f(X) \subset X$ be a continuous mapping, and let $h(X) = X$ be a homeomorphism. For $x \in X$, the set $\bigcup_{n=0}^{+\infty} f^n(x)$ is called the semi-orbit of $x$ under $f$ and the set $\bigcup_{n=0}^{+\infty} h^n(x)$ is called the orbit of $x$ under $h$. For $x \in X$, the closure of the semi-orbit of $x$ under $f$ is called the semi-orbit-closure of $x$ under $f$ and the closure of the orbit of $x$ under $h$ is called the orbit-closure of $x$ under $h$.

A nonvacuous subset $Y$ of $X$ is said to be semi-minimal (minimal) under $f(h)$ provided that the semi-orbit-closure (orbit-closure) of each point of $Y$ is $Y$. Clearly, any two semi-minimal (minimal) sets are either coincident or disjoint. It is easily proved that a subset $Y$ of $X$ is semi-minimal (minimal) under $f(h)$ if and only if $Y$ is nonvacuous, closed, $f(Y) \subset Y (h(Y) = Y)$, and furthermore $Y$ contains no proper subset with these properties. We follow Birkhoff [2, p. 198] in the terminology of "minimal set."

A decomposition of $X$ is defined to be a collection of nonvacuous closed pairwise disjoint subsets of $X$ which fill up $X$. We say that the mapping $f$ gives a semi-orbit-closure (a semi-minimal set) decomposition provided that the collection of semi-orbit-closures (semi-minimal sets) is a decomposition of $X$. Also, it is said that the homeomorphism $h$ gives an orbit-closure (a minimal-set) decomposition provided that the collection of orbit-closures (minimal sets) is a decomposition of $X$.

A point $x$ of $X$ is said to be almost periodic under $f$ provided that to each $\varepsilon > 0$ there corresponds a positive integer $N$ with the property that in every set of $N$ consecutive positive integers appears an integer $n$ such that $\rho(x, f^n(x)) < \varepsilon$. The mapping $f$ is said to be pointwise almost periodic provided that each point of $X$ is almost periodic under $f$. It is to be noted that various writers use the above terms in different senses and employ other terminologies for these notions.

**Lemma 1.** The mapping $f$ (homeomorphism $h$) gives a semi-orbit-closure (an orbit-closure) decomposition if and only if $f(h)$ gives a semi-minimal-set (a minimal-set) decomposition; and in either event, the two decompositions coincide.

Presented to the Society, August 14, 1944; received by the editors May 15, 1944 and November 15, 1944.

1 Numbers in brackets refer to the bibliography at the end of the paper.
The proof is easy and will be omitted.

**Lemma 2.** In order that the homeomorphism $h$ give an orbit-closure decomposition it is sufficient that $h$ give a semi-orbit-closure decomposition; and in case $X$ is compact, this condition is also necessary. In either event, the two decompositions coincide.

**Proof.** The proof of the sufficiency is easy and will be omitted. We establish the necessity. Let $C$ be an orbit-closure. By Lemma 1, it is enough to show that $C$ is a semi-minimal set. Let $Y$ be a nonvacuous closed subset of $C$ such that $h(Y) \subset Y$. The proof will be completed if we show $Y = C$. Define $Z = \prod_{n=0}^{\infty} h^n(Y)$. Now $Z$ is a nonvacuous closed subset of $C$ such that $h(Z) = Z$. Since $C$ is a minimal set by Lemma 1, $Z = C$ and, hence, $Y = C$.

**Lemma 3.** If $x \in X$ is almost periodic under $f$, then the semi-orbit-closure $C$ of $x$ is semi-minimal.

**Proof.** Suppose $C$ is not semi-minimal. Then there exists a point $y$ of $C$ such that the semi-orbit-closure $Y$ of $y$ is not $C$. Now $Y \subset C$. Also $x \in Y$, since otherwise $C \subset Y$ and thus $Y = C$. Let $2\varepsilon$ be the distance from $x$ to $Y$. There exists a positive integer $N$ such that in every set of $N$ consecutive positive integers appears an integer $n$ so that $\rho(x, f^n(x)) < \varepsilon$. Choose $\delta > 0$ so small that $z \in X$ with $\rho(y, z) < \delta$ implies $\rho(f^i(y), f^i(z)) < \varepsilon$ ($i = 1, 2, \ldots, N$). Now there exists a non-negative integer $p$ such that $\rho(y, f^p(x)) < \delta$. Also it is possible to find an integer $q$, $1 \leq q \leq N$, so that $\rho(x, f^{p+q}(x)) < \varepsilon$. Hence, $\rho(x, f^q(y)) < 2\varepsilon$ which is impossible.

**Lemma 4.** If $X$ is locally compact and if the subset $Y$ of $X$ is semi-minimal under $f$, then each point of $Y$ is almost periodic.

**Proof.** Assume some point $x$ of $Y$ is not almost periodic. There exists a neighborhood $U$ of $x$ such that $U$ is compact and such that for each positive integer $n$ there exists a point $x_n$ of $U \cdot Y$ with the property that $f^n(x_n) \in U$ $(m = 1, 2, \ldots, n)$. Some subsequence of $x_1$, $x_2$, $\ldots$ converges to some point, say $y$, of $U \cdot Y$. There exists a positive integer $M$ such that $f^M(y) \in U$ and, hence, there also exists a neighborhood $V$ of $y$ such that $f^M(V) \subset U$. Choose an integer $N$ so that $N > M$ and $x_N \in V$. Then, $f^M(x_N) \in U$ which is a contradiction.

**Theorem 1.** In order that the mapping $f$ give a semi-orbit-closure decomposition, it is sufficient that $f$ be pointwise almost periodic; and in case $X$ is locally compact, this condition is also necessary.

The proof follows easily from Lemmas 1, 3 and 4.
THEOREM 2. In order that the homeomorphism \( h \) give an orbit-closure decomposition it is sufficient that \( h \) be pointwise almost periodic; and in case \( X \) is compact, this condition is also necessary.

The proof proceeds easily from Theorem 1 and Lemma 2.


We pause in our main development to comment on the rôle of local compactness in the second part of Theorem 1. In this case the semi-orbit-closures are actually compact, as the following indicates.

THEOREM A. If \( x \in X \) is almost periodic under \( f \) and if there exists a neighborhood \( U \) of \( x \) whose closure is compact, then the semi-orbit-closure of \( x \) is itself compact.

Proof. There exists an integer \( N \) and a sequence \( \{ n_i \} \), \( i = 0, 1, \ldots \) of integers such that \( 0 = n_0 < n_1 < \cdots \), \( n_{i+1} - n_i \leq N \) and \( f^{n_i}(x) \in U \) for all \( i \). Define \( K = \bigcup_{r=0}^{N} f^r(U) \). Now \( K \) is compact. We show the semi-orbit of \( x \) is contained in \( K \), which completes the proof. Let \( n \) be any non-negative integer. There exists a non-negative integer \( i \) such that \( n_i \leq n \leq n_{i+1} \). Hence, \( f^n(x) = f^{n-n_i}f^{n_i}(x) \in f^{n-n_i}(U) \subseteq K. \)

THEOREM B. If \( Y \subseteq X \) is semi-minimal under \( f \) and if \( Y \) intersects a neighborhood \( U \) whose closure is compact (in particular, if \( X \) is locally compact), then \( Y \) is itself compact.

Proof. Let \( y \in Y \cdot U \). By the argument used in the proof of Lemma 4, it can be shown that \( y \) is almost periodic. The conclusion now follows from Theorem A. (Theorem B is not valid for a minimal set as the example of a discrete infinite orbit shows.)

Besicovitch [1] has constructed an interesting example of a homeomorphism of the plane onto itself which possesses some semi-orbits dense in the plane and which leaves the origin fixed. He seems to remark at the end of his paper (p. 65) that every semi-orbit, excluding the origin, is also dense in the plane. Theorem B would indicate that either this remark or our interpretation of it is in error. For, take \( X = Y \) to be the plane with the origin deleted. If every semi-orbit is dense in the punctured plane, then the punctured plane would be compact. Question: In Besicovitch's example [1], is the orbit of every point of the plane, excepting the origin, dense in the plane? We now continue with the main sequence of theorems.

Let \( \{ X_n \} \), \( n = 1, 2, \ldots \) be a sequence of subsets of \( X \). The set of all points \( x \) of \( X \) such that each neighborhood of \( x \) intersects \( X_n \) for...
almost all (infinitely many) positive integers \( n \) is denoted by 
\[
\lim \inf \{ X_n \} \quad \text{and} \quad \lim \sup \{ X_n \}.
\]
In case \( \lim \inf \{ X_n \} = \lim \sup \{ X_n \} \),
we denote this set by \( \lim \{ X_n \} \). Of course, for any sequence \( \{ X_n \} \),
\( \lim \inf \{ X_n \} \subseteq \lim \sup \{ X_n \} \).

Let \( D \) be a decomposition of \( X \). For \( x \in X \), let \( D(x) \) denote the element
of \( D \) containing \( x \). The decomposition \( D \) is said to be continuous
provided that \( x_0, x_n \in X \ (n = 1, 2, \cdots) \) with \( x_n \to x_0 \) implies \( D(x_0) \subseteq \lim \inf \{ D(x_n) \} \subseteq \lim \sup \{ D(x_n) \} \subseteq D(x_0) \).

Following G. A. Hedlund, we say the mapping \( f \) is uniformly pointwise almost periodic
provided that to each \( \varepsilon > 0 \) there corresponds a positive integer \( N \)
such that if \( x \in X \), then in every set of \( N \) consecutive positive integers appears an integer \( n \) so that 
\( \rho(x, f^n(x)) < \varepsilon \). Clearly, if \( f \) is uniformly pointwise almost periodic, then \( f \) is pointwise
almost periodic.

**Lemma 5.** For \( x \in X \), let \( C(x) \) denote the semi-orbit-closure of \( x \) under \( f \).
If \( x_0, x_n \in X \ (n = 1, 2, \cdots) \) with \( x_n \to x_0 \), then \( C(x_0) \subseteq \lim \inf \{ C(x_n) \} \).

**Proof.** Let \( x \in C(x_0) \) and let \( U \) be any neighborhood of \( x \). For some
non-negative integer \( k, f^k(x_0) \in U \). By the continuity of \( f^k, f^k(x_0) \in U \)
for almost all positive integers \( n \), that is, \( U \) intersects \( C(x_0) \) for almost
all positive integers \( n \). The conclusion follows.

**Theorem 3.** In order that the mapping \( f \) give a continuous semi-orbit-closure decomposition
it is sufficient that \( f \) be uniformly pointwise almost periodic; and in case \( X \) is compact, this condition is also necessary.

**Proof.** We establish the sufficiency. Let \( D \) denote the collection
of semi-orbit-closures. By Theorem 1, \( D \) is a decomposition of \( X \).
By Lemma 5, it is enough to prove that \( \lim \sup \{ C(x_n) \} \subseteq C(x_0) \)
for \( x_0, x_n \in X \ (n = 1, 2, \cdots) \) with \( x_n \to x_0 \), where \( C(x) \) denotes the semi-orbit-closure of the point \( x \). Assume this is false. Then there exist points
\( x_0, x_n \in X \ (n = 1, 2, \cdots) \) such that \( x_n \to x_0 \) and \( \lim \sup \{ C(x_n) \} \subsetneq C(x_0) \). Thus there exist a point \( x \) of \( X \), a monotone increasing sequence \( n_1, n_2, \cdots \) of positive integers, and a sequence \( m_1, m_2, \cdots \) of
non-negative integers such that \( f^{m_i(x_{n_i})} \to x \) and \( x \in C(x_0) \). Hence,
\( C(x) \cdot C(x_0) = \varnothing \) and \( x_0 \in C(x) \). Let \( 2\varepsilon \) denote the distance from \( x_0 \) to \( C(x) \).
Since \( f \) is uniformly pointwise almost periodic, there exists a positive integer \( k \) such that for each positive integer \( i \) it is possible to
find an integer \( k_i \) with the properties that \( 1 \leq k_i \leq k \) and 
\( \rho(x_{n_i}, f^{m_i+k_i(x_{n_i})}) < \varepsilon \). There exists an integer \( k_0 \) such that \( k_i = k_0 \)
for infinitely many positive integers \( i \). Since also \( x_{n_i} \to x_0 \) and \( f^{m_i+k_0(x_{n_i})} \to f^{k_0}(x) \), we have 
\( \rho(x_0, f^{k_0}(x)) \leq \varepsilon < 2\varepsilon \). Hence, the distance from \( x_0 \) to \( C(x) \) is less than \( 2 \varepsilon \). This is a contradiction.
We establish the necessity. Suppose \( f \) is not uniformly pointwise almost periodic. Then there exists a positive number \( \epsilon \) such that for each positive integer \( n \) it is possible to find a positive integer \( m_n \) and a point \( x_n \) of \( X \) such that \( N_\epsilon(x_n) \cdot \sum_{i=0}^{m_n} f^{m_n+i}(x_n) = \Lambda \), where \( N_\delta(y) \) denotes the \( \delta \)-neighborhood of the point \( y \). We may suppose that \( x_n \to x_0 \) and \( f^{m_n}(x_n) \to x \) for some points \( x_0, x \) of \( X \). For all sufficiently large positive integers \( n \), \( N_{\epsilon/2}(x_0) \cdot \sum_{i=0}^{m_n} f^{m_n+i}(x_n) = \Lambda \). It follows that \( N_{\epsilon/2}(x_0) \cdot C(x) = \Lambda \) and \( x_0 \in C(x) \), where \( C(y) \) denotes the semi-orbit-closure of the point \( y \). Now \( \lim \{ C(x_n) \} = C(x_0) \) and \( C(x) = \lim \{ C(f^{m_n}(x_n)) \} = \lim \{ C(x_n) \} \). Hence, \( C(x_0) = C(x) \) and \( x_0 \in C(x) \). This is a contradiction.

**Theorem 4.** In order that the homeomorphism \( h \) give a continuous orbit-closure decomposition it is sufficient that \( h \) be uniformly pointwise almost periodic; and in case \( X \) is compact, this condition is also necessary.

The proof follows readily from Theorem 3 and Lemma 2.

It is worthy of note that if \( X \) is compact, then there exists a subset \( Y \) of \( X \) such that \( f(Y) = Y \) is uniformly pointwise almost periodic. The proof is short. The property \( P \) of being a nonvacuous closed subset \( Z \) of \( X \) such that \( f(Z) \subseteq Z \) is easily shown to be inducible. By the Brouwer reduction theorem, there exists a subset \( Y \) which has property \( P \) irreducibly. Then \( Y \) is semi-minimal and \( f(Y) = Y \), since \( f(Y) \) has property \( P \). The conclusion now follows from Theorem 3.

**Bibliography**